Some Model Theoretic Results in Set Theory

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Contents

1 Introduction

In this thesis we are going to explore the absoluteness results in set theory from a model theoretic view and then apply these results to find model companion of set theory. Roughly speaking, a property is absolute if it doesn't change its truth value among some classes. And the model companion of a theory is another theory that has the exact universal part of the original theory and has a nice property called model complete. This thesis only assume readers to know the most basic notions in logic, e.g. first-order language, first-order structure and the proof system of deduction. You can refer to section 1 if you want to review concepts in set theory and model theory.

Assuming the consistency of ZFC, our ultimate goal is to investigate the properties of the model companion of set theory. The model companion of a theory is another theory that is model complete and has the exact universal part of the original theory. This is a concept generalized from structures in algebra, for example, algebraic closed fields.

It is not accurate to say "the model companion of set theory", since we are considering ZFC in an extended language, with each relation symbols and function symbols definable by formulas. Throughout this thesis, though, we'll use this inaccurate expression when we discuss the contents of theorems without mentioning the theorems explicitly.

There might be other ways to deal with metatheory, but in this thesis, we will take the view point of embedding model theory into set theory, i.e. to view a symbol as a set, a language as a set, a function as a set, a structure as a set, a theory as a set, etc. This is to keep the minimal amount of requirements over metatheory. In this way, we only need to deal with several nonlogical symbols in metatheory since the language of set theory only contains one additional symbol $\in$.

Since ZFC can be described in a finite way, we can express ZFC inside set theory by a formula, denoted as $\sharp(ZFC)$.

By induction in metatheory, if we have a theorem in set theory proving $ZFC \vdash \phi$, we can get $ZF \vdash (\sharp(ZFC) \vdash \sharp\phi)$, where $\sharp\phi$ is the expression of ϕ inside set theory.

Gödel's Completeness theorem states that

$$ZF \vdash \forall T(T \text{ is a set of formulas} \to (Con(T) \leftrightarrow Sat(T))),$$

where $Con(T)$ is "T is consistent", i.e. there is no contradiction derived from T, and $Sat(T)$ is "T is satisfiable", i.e T has a model. By Gödel's Completeness Theorem, $ZF \vdash Con(ZFC) \to Sat(ZFC)$, so we can always talk about set models of ZFC if we assume $Con(ZFC)$. And we can talk about set models of ZFC satisfying certain theorem in set theory in this way.

Throughout this thesis we are going to assume $Con(ZFC)$, then ZFC becomes a consistent theory, and we can talk about the consistent theories that extend ZFC, and the model companion of ZFC (in some extended language), which is another theory that has the exact same universal part of ZFC (in some extended language), and has a nice property called model completeness. We'll also investigate different absoluteness theorems in set theory from a model theoretic point of view throughout the way, and discuss the relation of CH with model companion of ZFC (in some extended language).

In this view point, all theorems we prove related to model companion of set theory will be of the form $Con(ZFC) \to \sigma$, where σ is our statement in the theorem.

We already know that CH is neither provable nor disprovable from ZFC, i.e.

$$ZFC \nvdash CH \text{ and } ZFC \nvdash \neg CH.$$

A natural question to ask here is that what is the relation between CH and the model companion of set theory. We investigate this question in Section 4.3.

We briefly recap the basic notions in set theory and model theory in Section 1, then we introduce basic results in descriptive set theory in Section 2.1, involving constructing universal set for $\mathbf{\Sigma}^1_n$ sets, and prove $\mathbf{\Pi}^1_n \setminus \mathbf{\Sigma}^1_n$ is nonempty.

In Section 2.2, we will show how to interpret H_{ω_1} in 2nd-order arithmetic. This is done by coding each set in H_{ω_1} using a relation $R \subseteq \omega \times \omega$ that collapses onto it. In particular, to match the results in descriptive set theory, we convert relation R into element of ω^ω and check the complexity of this interpretation.

Section 2.3 and Section 2.4 are standard results in preparation for Section 4.

In Section 3, we investigate three Absoluteness Theorems in set theory, from a model theoretic point of view. In particular, Mostowski's Absoluteness Theorem is a weaker version of Shoenfield's Absoluteness Theorem, which turns out to be enough to prove the correspondence of $\mathbf{\Sigma}_n$ in H_{ω_1} and $\mathbf{\Sigma}^1_{n+1}$, so we include it here. Then we prove a version of Levy's absoluteness Theorem in a model theoretic point of view. We also include two interesting results that can be proved from these Absoluteness Theorems.

Then we investigate the model companion of set theory under different levels of partial Morleyizations. In Theorem 4.1.2, we show that partial Morleyized to Δ_0 level cannot guarantee there is a model companion of set theory. So we add reasonable relation symbols into our language for definable sets to get a model companion of set theory, see Theorem 4.2.1. Finally, we investigate the relation between CH and the model companion of set theory, see Theorem 4.3.1.

The main results in this thesis are heavily based on [Viale, 2021], [Venturi and Viale, 2022a] and [Venturi and Viale, 2022b].

Notations

We introduce the following notations for this essay and facts related with these concepts. More details can be found in any standard logic books.

Set Theory

We denote the language of set theory as $\{\in\}$, which contains equality $=$, one predicate symbol $\in$ and no constant or function symbols. Since we use concepts in model theory to study set theory, it is necessary to specify languages. We sometimes abbreviate "in the language of set theory" as "in set theory".

ZF are **axioms of Zermelo-Fraenkel set theory**. See for example[Kunen, 1980, Page xv]. We use ZF^- to denote $ZF-$ Power Set, ZFC^- to denote ZF^-+ Choice, ZFC to denote $ZF+$ Choice. We'll work on ZFC^- mostly afterwards.

$trcl(x)$ is the **transitive closure** of x, see for example[Kunen, 1980, Page 99], $|x|$ is the cardinality of x, and we denote

$$H_\kappa = \{x : |trcl(x)| < \kappa\}.$$

$rank(x)$ is the **rank function** defined by $rank(x) = sup\{rank(y) : y \in x\}$.

If κ is an ordinal, $\kappa + 1$ denotes the successor ordinal of κ and κ^+ denotes the successor cardinal of κ.

B^A is the set (or class) of all functions $f : A \to B$.

Suppose R is a binary relation, then $dom(R) = \{x : \exists y(xRy)\}$, $range(R) = \{y : \exists x(xRy)\}$, $field(R) = dom(R) \cup range(R)$.

Suppose f is a function, then we define the graph of a function f to be $\{\langle x, y\rangle : f(x) = y\}$. We define $f \restriction A$ to be f restricted to $A \cap dom(f)$.

If $\phi(x)$ is a formula in set theory with one free variable, a class defined by this formula is $\{x : \phi(x)\}$, i.e. it is all the sets that satisfy this formula. This is to deal with Russell's Paradox, which states there is no set that contains all sets. We will call a class proper if it is not a set.

Model Theory

A (τ-)**theory** is a set of sentences in language τ which is consistent and closed under logical implications.

For two set of τ-sentences A, B, we use $A + B$ to denote $A \cup B$. If $A \subseteq B$ and $A \models B$, we say that B is axiomatized by A.

For any set of sentences Σ which is consistent, we define the consequence of Σ as

$$Cn(\Sigma) = \{\sigma : \sigma \text{ is a sentence and } \Sigma \models \sigma\}.$$

Fact 1.0.1. *(i)For a consistent set of sentences Σ, $Cn(\Sigma)$ is the smallest theory that contains Σ, i.e. for any theory $T \supseteq \Sigma$, we have $T \supseteq Cn(\Sigma)$.*
(ii)For consistent sets of sentences Σ_1, Σ_2, if $\Sigma_1 \models \Sigma_2$, then $Cn(\Sigma_1) \supseteq Cn(\Sigma_2)$.

We assume the readers to know Gödel's Completeness Theorem, hence we don't distinguish $\models$ and $\vdash$, consistent and satisfiable here.

We say a sentence σ is logically true if $\models \sigma$.

If A is a set of sentences, σ, δ are two sentences, we say $\sigma \equiv \delta$ under A if $A \vdash \sigma \leftrightarrow \delta$.

A τ-structure is a structure in language τ.

$\vec{a}$ denotes a tuple of arbitrary length. For convenience, we drop the arrow and write a when it is clear, e.g. we use $\psi(a)$ to denote $\psi(\vec{a})$. And when f is a function defined on A and $\vec{a} = \langle a_1, ..., a_n \rangle \in A^n$, we use $f(\vec{a})$ to denote $\langle f(a_1), ..., f(a_n) \rangle$.

If $\mathfrak{M}$ is a τ-structure, we use $|\mathfrak{M}|$ to denote the domain of $\mathfrak{M}$. When it is clear, we also use $\mathfrak{M}$ to denote the domain of the structure.

We denote

$$Th(\mathfrak{M}) = \{\sigma : \sigma \text{ is a sentence and } \mathfrak{M} \models \sigma\}.$$

For a set $\mathcal{A}$ of structures, we denote

$$Th(\mathcal{A}) = \{\sigma : \sigma \text{ is a sentence and } \forall \mathfrak{M} \in \mathcal{A}(\mathfrak{M} \models \sigma)\}.$$

Fact 1.0.2. *$Th(\mathfrak{M})$, $Th(\mathcal{A})$ defined above are theories. Thus, we call $Th(\mathfrak{M})$ the theory of structure $\mathfrak{M}$ and $Th(\mathcal{A})$ the theory of set $\mathcal{A}$ of structures.*

For two structures $\mathfrak{M}, \mathfrak{N}$, if $\mathfrak{M} \sqsubseteq \mathfrak{N}$, we call $\mathfrak{M}$ a **substructure** of $\mathfrak{N}$, $\mathfrak{N}$ a superstructure of $\mathfrak{M}$.

We say $\mathfrak{M} \prec \mathfrak{N}$ if $\mathfrak{M} \sqsubseteq \mathfrak{N}$, and for any formula $\phi(\vec{x})$ and $\vec{a} \in \mathfrak{M}$, we have

$$\mathfrak{M} \models \phi(\vec{a}) \leftrightarrow \mathfrak{N} \models \phi(\vec{a}).$$

In this case, we call $\mathfrak{M}$ an **elementary substructure** of $\mathfrak{N}$.

We say $\mathfrak{M}$ is a model of theory T if $\mathfrak{M} \models T$.

For τ a language with k_λ-ary relation symbols R_λ, k_β-ary function symbols and constant symbols c_γ, a τ-structure $\mathfrak{M}$ with domain M, and $N \subseteq M$, we say $\mathfrak{M}$ restricted to N is a structure if

$$\mathfrak{N} := (N; R_\lambda^{\mathfrak{M}} \cap N^{k_\lambda}, f_\beta^{\mathfrak{M}} \cap N^{k_\beta+1}, c_\gamma^{\mathfrak{M}})$$

is a structure, i.e. $f_\beta^{\mathfrak{M}} \cap N^{k_\beta+1}$ is a function from N^{k_β} to N and $c_\gamma^{\mathfrak{M}} \in N$ for each β, γ. In this case, we say $\mathfrak{N}$ is $\mathfrak{M}$ restricted to domain N.

$Diag(\mathfrak{M})$ is the diagram of $\mathfrak{M}$ in language τ, see for example[Tent and Ziegler, 2012, Page 11].

2 Background Knowledge

2.1 Descriptive Set Theory

Descriptive set theory is the study of sets that are definable in 2nd-order arithmetic. When we talk about 2nd-order arithmetic, we always mean the 2nd-order structure $(\omega, \omega^\omega; +, \times, 0, 1, <, Exp)$ interpreted in set theory, and by a formula in 2nd-order arithmetic we mean a formula in this 2nd-order language interpreted in set theory, with small letters $x, y, ...$ for 1st-order parameters and capital letters $X, Y, ...$ for 2nd-order parameters. See [Enderton, 2002, section 2.7,4.1] for details. We say a set A is definable in 2nd-order arithmetic if there is a formula $\psi(\vec{x}, \vec{Y})$ in 2nd-order arithmetic that

$$(\vec{x}, \vec{Y}) \in A \leftrightarrow (\omega, \omega^\omega; +, \times, 0, 1, <, Exp) \models \psi(\vec{x}, \vec{Y}).$$

We'll define the complexity of definable sets in 2nd-order arithmetic using the complexity of formulas defining them. Then we prove some basic facts of definable subsets of ω^ω. This section is based on [Jech, 2006, Page 479–511].

We define the complexity of formulas in 2nd-order arithmetic.

Definition 2.1.1. *(i)A formula in 2nd-order arithmetic is Σ_0^0 if it is logically equivalent to a formula with only bounded first-order quantifiers ($\forall x < y, \exists x < y$).*
(ii)A formula is Π_n^0 if it is logically equivalent to a formula of form $\neg\psi(\vec{x})$ where ψ is Σ_n^0.
(iii)A formula is Σ_{n+1}^0 if it is logically equivalent to a formula of form $\exists x \in \omega\psi(x, \vec{y})$, where ψ is Π_n^0.

Definition 2.1.2. *(i)A formula in 2nd-order arithmetic is* Σ_0^1 *if it is logically equivalent to a* Σ_n^0 *or* Π_n^0 *formula.*
(ii)A formula is Π_n^1 *if it is logically equivalent to a formula of form* $\neg\psi(\vec{x})$ *where* ψ *is* Σ_n^1.
(iii)A formula is Σ_{n+1}^1 *if it is logically equivalent to a formula of form* $\exists x \in \omega^\omega \psi(x, \vec{y})$, *where* ψ *is* Π_n^1.

Then we define the complexity of definable subsets of ω^ω using the complexity of formulas defining them.

Definition 2.1.3. *We say* $A \subseteq \omega^\omega$ *is* Σ_n^i $(i = 0, 1)$ *if it can be defined by a* Σ_n^i *formula* $\psi(X)$ *without parameter, i.e.*

$$X \in A \leftrightarrow \psi(X).$$

Similarly we say a subset of ω^ω *is* Π_n^i $(i = 0, 1)$ *if it can be defined by a* Π_n^i *formula without parameter.*
For $a \in \omega^\omega$, *we say* $B \subseteq \omega^\omega$ *is* $\Sigma_n^i(a)$ $(i = 0, 1)$ *if there is a* Σ_n^i *formula* $\psi(X, Y)$ *s.t.*

$$X \in B \leftrightarrow \psi(X, a).$$

We can define $\Pi_n^i(a)$ *set similarly.*
We define $\boldsymbol{\Sigma}_n^i, \boldsymbol{\Pi}_n^i$ *as*

$$\boldsymbol{\Sigma}_n^i = \cup_{a \in \omega^\omega} \Sigma_n^i(a),$$
$$\boldsymbol{\Pi}_n^i = \cup_{a \in \omega^\omega} \Pi_n^i(a).$$

If B *is both* Σ_n^i *and* Π_n^i, *we say it is* Δ_n^i. *Similarly for* $\Delta_n^i(a)$, $\boldsymbol{\Delta}_n^i$.

Remark 2.1.1. *Similarly, we can define the complexity of definable subsets of* $\omega, \omega^n, (\omega^\omega)^n$ *etc using the complexity of formulas defining it.*

Definition 2.1.4. *We say* A *is recursive if it is* Δ_1^0. *We say a function is recursive if its graph (which is a subset of* ω^2*) is recursive.*

Remark 2.1.2. *The concept of recursive subset of* ω *defined by Turing machines is equivalent to this definition in arithmetic.*

Remark 2.1.3. *For how adding quantifiers, taking* $\wedge$, $\vee$ *will change the complexity of the formulas, see [Jech, 2006, Page 480]. For example, if* $\psi(X, \vec{Y})$ *is* Σ_n^1, $\exists X \in \omega^\omega \psi(X, \vec{Y})$ *is also* Σ_n^1.

Remark 2.1.4. *Instead, we can take our 2nd-order arithmetic to be* $(\omega, \omega^\omega; +, \times, 0, 1)$ *and define the complexity of definable sets accordingly. This will not change the complexity of sets definable in 2nd-order arithmetic up to* Δ_1 *level. We add* $<, Exp$ *to our structure so that we can code finite sequences of natural numbers in a simple way.*

Definition 2.1.5. *We have the following notations:*

$$Seq(K) = \bigcup_{n \in \omega} K^n,$$

where K^n is the set of all functions from n into K. We also use $K^{<\omega}$ to denote $Seq(K)$ sometimes.
We define $Seq = Seq(\omega)$.

Definition 2.1.6. *If $s \in Seq(K)$, we define $[s] = \{X \in K^\omega : X \upharpoonright dom(s) = s\}$.*

Definition 2.1.7. *We define $\mathcal{N}$ as ω^ω with the topology generated by $\{[s] : s \in Seq\}$. $\mathcal{N}^k$ is $(\omega^\omega)^k$ with the product topology.*

Definition 2.1.8. *We fix a recursive enumeration $\Gamma : \omega \to Seq$, which means Γ is a surjective recursive function and functions $dom(\Gamma(n)) = m$(with input n and output m),$\Gamma(n)(k) = i$ are recursive. We denote s_n to be $\Gamma(n)$ afterwards.*

Remark 2.1.5. *This can be done by Gödel coding. Then we can view Seq as ω and talk about recursive formulas/subsets defined on Seq.*

We state here without proving Kleene normal form for Σ^1_1 formulas[Simpson, 2009].

Theorem 2.1.1. *(Kleene normal form for Σ^1_1 formulas) Suppose $\psi(X)$ is Σ^1_1, where $X \in \mathcal{N}$, then there exists Σ^0_0 formula $\phi(X, Y)$ with $X, Y \in \mathcal{N}$ such that*

$$\psi(X) \equiv \exists Y \forall n \in \omega \phi(X \upharpoonright n, Y \upharpoonright n).$$

Lemma 2.1.1. *A is $\boldsymbol{\Sigma}^1_1$ if and only if it is the projection of a closed set in $\mathcal{N}^2$.*

Proof. If A is defined by $\psi(X, a)$, where $a \in \mathcal{N}$, from Theorem 2.1.1,

$$\psi(X, a) \equiv \exists Y \forall n \phi(X \upharpoonright n, Y \upharpoonright n, a \upharpoonright n).$$

$\{(X, Y) \in \mathcal{N}^2 : \forall n \phi(X \upharpoonright n, Y \upharpoonright n, a \upharpoonright n)\}$ is a closed set.
If A is the projection of a closed set, $X \in A \leftrightarrow \exists Y\ (X, Y) \in B$, where B is closed. There exists $I \subseteq \mathcal{N}^2$ such that $B^c = \cup_{\langle s,t\rangle \in I}[s] \times [t]$. Let $a \in \mathcal{N}$ code I by $a(2^m \cdot 3^n) = 0$ iff $\langle s_m, s_n\rangle \in I$. Let ϕ be for all $x, y, a \in Seq$,

$$\phi(x, y, a) \equiv \exists m, n \in \omega[(x \upharpoonright dom(s_m) = s_m) \wedge (y \upharpoonright dom(s_n) = s_n) \wedge a(2^m \cdot 3^m) = 0].$$

Then $X \in A$ iff $\exists Y \forall n\ \phi(X \upharpoonright n, Y \upharpoonright n, a \upharpoonright n)$. A is $\boldsymbol{\Sigma}^1_1$. ☐

Lemma 2.1.2. *For every n, there is a universal $\boldsymbol{\Sigma}^1_n$ set in $\mathcal{N}^2$.*

Proof. From Lemma 2.1.1, $U_1(X, a) \equiv \exists Y \forall n\ \phi(X \upharpoonright n, Y \upharpoonright n, a \upharpoonright n)$ is a universal $\boldsymbol{\Sigma}^1_1$ set.
If there is a $\boldsymbol{\Sigma}^1_n$ set $U_n(X, a)$, for $X, Y \in \mathcal{N}$, let $h(X, Y)(n) = 2^{X(n)} \cdot 3^{Y(n)}$. $U_{n+1}(X, a) = \exists Y \neg U_n(h(X, Y), a)$ is $\boldsymbol{\Sigma}^1_{n+1}$ and for any $\boldsymbol{\Sigma}^1_{n+1}$ set A, there exists $\boldsymbol{\Sigma}^1_n$ set $B \subseteq \mathcal{N}^2$,

$$X \in A \leftrightarrow \exists Y \neg (X, Y) \in B$$

Let $(X, Y) \in B$ iff $h(X, Y) \in C$, C is $\boldsymbol{\Sigma}^1_n$, there exists $a \in \mathcal{N}$ such that

$$Z \in C \leftrightarrow U_n(Z, a).$$

Thus,

$$X \in A \leftrightarrow \exists Y \neg C(h(X, Y)) \leftrightarrow \exists Y \neg U_n(h(X, Y), a) \leftrightarrow U_{n+1}(X, a).$$

U_{n+1} is a universal $\boldsymbol{\Sigma}^1_{n+1}$ set. ☐

Corollary 2.1.1. *For any $n \in \omega$, there is a set in $\mathbf{\Sigma}^1_n \setminus \mathbf{\Pi}^1_n$.*

Proof. Let U be a universal $\mathbf{\Sigma}^1_n$ set in $\mathcal{N}^2$, $A = \{X \in \mathcal{N} : (X, X) \in U\}$. Then A is $\mathbf{\Sigma}^1_n$. If A is $\mathbf{\Pi}^1_n$, there is a such that $A^c = \{X \in \mathcal{N} : (X, a) \in U\}$, then $(a, a) \in U \leftrightarrow a \in A \leftrightarrow a \notin A$, contradiction. $\square$

2.2 Interpret H_{ω_1} in 2nd-order arithmetic

When we talk about 2nd-order arithmetic $(\omega, \omega^\omega; +, \times, 0, 1, <, Exp)$, we are always referring to the interpretation of it in set theory. For example, we say a formula is in 2nd-order arithmetic if it is a formula in set theory with only two sorts of quantifiers $\forall x \in \omega$, $\forall X \in \omega^\omega$, and $+, \times, 0, 1, <, Exp$ as their standard definitions in set theory.

We want to show the correspondence between 2nd-order arithmetic and hereditarily countable sets $(H_{\omega_1}, \in)$. These two structures are bi-interpretable[], i.e. both structures can be interpreted in the other and know the combination of interpretations. The interpretation of 2nd-order arithmetic in $(H_{\omega_1}, \in)$ is the natural interpretation, i.e. ω is $\omega^{H_{\omega_1}}$, etc. Since H_{ω_1} is transitive model of ZFC^-, these concepts are absolute. We will only show the interpretation of $(H_{\omega_1}, \in)$ in 2nd-order arithmetic here.

We want to code $a \in H_{\omega_1}$ using $R \subseteq \omega^2$. Note that let $\kappa = |trcl(\{a\})|$, then $\kappa \leq \omega$ and there is a bijection $f : trcl(\{a\}) \to \kappa$ such that $f(a) = 0$. And a is the "top" element of $trcl(\{a\})$. We will show that for binary relation $R \subseteq \omega^2$ well-founded, extensional on $field(R)$ and has a "top" element 0, there is $a \in H_{\omega_1}$ such that the Mostowski's collapse[Jech, 2006, Page 69] of $(field(R), R)$ is $(trcl(\{a\}), \in)$. So R codes a.

Definition 2.2.1. *We write a function $f : (A, R) \to (B, S)$ if f is a function from A to B, $R \subseteq A^2$, $S \subseteq B^2$, and for all $x, y \in A$,*

$$\langle x, y\rangle \in R \text{ if and only if } \langle f(x), f(y)\rangle \in S.$$

We first define "top" element for a relation.

Definition 2.2.2. *Given a binary relation R on A, we say a is a top element of (A, R) if $a \in A$ and for all $b \in A$, there exists $n \in \omega$ and a function $f : n+1 \to A$ such that for all $i < n$, $f(i+1)Rf(i)$, $f(0) = a$ and $f(n) = b$.*

Remark 2.2.1. *It is obvious that isomorphism preserves top element.*

Lemma 2.2.1. *If R is well-founded on A, there is at most one top element of (A, R).*

Proof. If not, let a_1, a_2 be top element of (A, R) and $a_1 \neq a_2$. By definition, there exists $n_1, n_2 \in \omega$ and function $f : n_1 + n_2 + 1 \to A$ such that $f(0) = a_1$, $f(n_1) = a_2$, $f(n_1 + n_2) = a_1$ and for all $i < n_1 + n_2$, $f(i + 1)Rf(i)$.

This contradicts well-foundedness since $range(f)$ does not have an R-minimal element. $\square$

We define "correct" $R \subseteq \omega^2$ that there is some $a \in H_{\omega_1}$ s.t $(field(R), R)$ is isomorphic to $(trcl(\{a\}), \in)$.

Definition 2.2.3. *We define $WF \subseteq \mathcal{P}(\omega^2)$, an equivalence relation $\sim$ on WF, and a binary relation E on WF as follows:*
$WF = \{R \subseteq \omega^2 : R$ is well-founded, extensional on $field(R)$ and 0 is a top element of $(field(R), R)\}$.
For $R, S \in WF$, $R \sim S$ if $(field(R), R) \cong (field(S), S)$.
For $R, S \in WF$, RES if there exists a function $f : (field(R), R) \to (field(S), S)$ that is $1-1$, $f(0)S0$ and

$$\forall n \in \omega[(\exists m \in \omega(nSf(m))) \to \exists k \in \omega(f(k) = n)].$$

Remark 2.2.2. *It is obvious that $\sim$ is an equivalence relation on WF. Thus, we can define*

$$WF/\sim = \{[R]_\sim : R \in WF\}$$

with $[R]_\sim = \{S \in WF : R \sim S\}$, and a binary relation $E/\sim$ on $WF/\sim$ to be

$$\langle [R]_\sim, [S]_\sim \rangle \in E/\sim \text{ if and only if } \langle R, S \rangle \in E.$$

Since isomorphism preserves top element, it is easy to check $E/\sim$ is well-defined.

Definition 2.2.4. *Let R is a well-founded, extensional binary relation on $field(R)$, we define Π_R to be for every $x \in field(R)$,*

$$\Pi_R(x) = \{\Pi_R(y) : yRx\}.$$

Remark 2.2.3. *From Mostowski's Collapsing Theorem[Jech, 2006, Page 69], Π_R is well-defined, and an isomorphism from $(field(R), R)$ to $(b, \in)$ with some transitive set b.*

Lemma 2.2.2. *If $R \in WF$, let $\Pi_R(0) = a$, then $range(\Pi_R) = trcl(\{a\})$.*

Proof. Since $range(\Pi_R)$ is transitive and $a \in range(\Pi_R)$, $trcl(\{a\}) \subseteq range(\Pi_R)$. For any $b \in range(\Pi_R)$, there is $n \in \omega$ and a function $f : n+1 \to rangle(\Pi_R)$ such that for all $i < n$, $f(i+1) \in f(i)$, $f(0) = a$ and $f(n) = b$, hence $b \in trcl(\{a\})$. $\square$

Lemma 2.2.3. *If a, b are transitive, $f : (a, \in) \to (b, \in)$ is an isomorphism, then f is the identity map.*

Proof. Suppose not, let $\gamma = min\{rank(c) : c \in a$ and $f(c) \neq c\}$. Take $d \in a$ such that $rank(d) = \gamma$. Then

$$f(d) = \{m : m \in f(d)\} = \{f(c) : f(c) \in f(d)\} = \{c : c \in d\} = d,$$

contradiction. $\square$

Theorem 2.2.1. *Let* $Code : (WF/\sim, E/\sim) \to (H_{\omega_1}, \in)$ *be defined by*

$$Code([R]_\sim) = \Pi_R(0),$$

then Code is well-defined and an isomorphism.

Proof. For $R, S \in WF$, if $R \sim S$, there is an isomorphism $\sigma : (field(R), R) \to (field(S), S)$. Since isomorphism preserves top elements, $\sigma(0) = 0$. $\Pi_S \circ \sigma \circ \Pi_R^{-1}$ is an isomorphism, by Lemma 2.2.3, it is the identity map, and

$$\Pi_R(0) = \Pi_S \circ \sigma \circ \Pi_R^{-1}(\Pi_R(0)) = \Pi_S(0).$$

Thus, $Code$ is well-defined.
For any $a \in H_{\omega_1}$, there is $\kappa \leq \omega$ and an isomorphism $f : trcl(\{a\}) \to \kappa$ such that $f(a) = 0$. f induced a binary relation $R \subseteq \kappa^2$ such that mRn if and only if $f^{-1}(m) \in f^{-1}(n)$. It is easy to check $R \in WF$. $\Pi_R \circ f$ is an isomorphism between $(trcl(\{a\}), \in)$ and $(range(\Pi_R), \in)$, by Lemma 2.2.3, $\Pi_R \circ f$ is identity, and $\Pi_R(0) = \Pi_R(f(a)) = a$. Thus, $Code$ is onto.
For $R, S \in WF$, if $\Pi_R(0) = \Pi_S(0)$, by Lemma 2.2.2, $range(\Pi_R) = range(\Pi_S)$, hence $\Pi_S^{-1} \circ \Pi_R$ is an isomorphism between $(field(R), R)$ and $(field(S), S)$. Thus, $Code$ is $1-1$.
For $R, S \in WF$, if RES, there exists a function $f : (field(R), R) \to (field(S), S)$ that is $1-1$, $f(0)S0$ and

$$\forall n \in \omega[(\exists m \in \omega(nSf(m))) \to \exists k \in \omega(f(k) = n)].$$

Claim. For all $x \in range(\Pi_R)$, $\Pi_S \circ f \circ \Pi_R^{-1}(x) = x$.
By induction on $(range(\Pi_R), \in)$, $\Pi_S \circ f \circ \Pi_R^{-1}(x) = \{\Pi_S(z) : zS(f \circ \Pi_R^{-1}(x))\} = \{\Pi_S(f(k)) : kR\Pi_R^{-1}(x)\} = \{\Pi_S \circ f \circ \Pi_R^{-1}(y) : y \in x\} = x.\square$
$\Pi_R(0) = Pi_S \circ f \circ \Pi_R^{-1}(\Pi_R(0)) = \Pi_S(f(0))$ and $\Pi_S(f(0)) \in \Pi_S(0)$, thus $\Pi_R(0) \in \Pi_S(0)$.
For $R, S \in WF$, if $\Pi_R(0) \in \Pi_S(0)$, let

$$I : (trcl(\{\Pi_R(0)\}), \in) \to (trcl(\{\Pi_S(0)\}), \in)$$

be identity on $trcl(\{\Pi_R(0)\})$, then we can check that $\Pi_S^{-1} \circ I \circ \Pi_R$ satisfies the condition on f in the definition of E, hence RES. $\square$

Remark 2.2.4. *Similarly, we can code elements of* H_{κ^+} *using* $R \subseteq \kappa^2$.

Remark 2.2.5. *Our ultimate goal is to code elements of* H_{ω_1} *using* $X \in \omega^\omega$ *with low complexity in 2nd-order arithmetic* $(\omega, \omega^\omega; +, \times, 0, 1, <, Exp)$. *We'll show how to do it below.*
Let $G : \omega^2 \to \omega$ *be function* $G(m, n) = 2^m \cdot 3^n$.
For $X \in \omega^\omega$, *let* $R_X \subseteq \omega^2$ *be*

$$\langle m, n\rangle \in R_X \text{ if and only if } X(G(m, n)) = 0.$$

Let $\mathcal{R} : \omega^\omega \to \mathcal{P}(\omega^2)$, *where* $\mathcal{R}(X) = R_X$. $\mathcal{R}$ *is a surjection.*
It is easy to check that for $X, Y \in \omega^\omega$, $R_X \in WF$ *is* Π^1_1, $R_X \sim R_Y$ *is* Σ^1_1, $R_X E R_Y$ *is* Σ^1_1.

2.3 Model companionship

We work in language τ in this subsection. We first define the complexity of τ-formulas in model theory.

Definition 2.3.1. *(i)A τ-formula is Σ_0 if it is logically equivalent to a quantifier-free τ-formula.*
(ii)A τ-formula is Π_n if it is logically equivalent to a τ-formula of form $\neg\psi(\vec{x})$ where ψ is Σ_n.
(iii)A τ-formula is Σ_{n+1} if it is logically equivalent to a τ-formula of form $\exists\vec{x}\psi(\vec{x},\vec{y})$, where ψ is Π_n.

Definition 2.3.2. *If a τ-formula is both Σ_n and Π_n, we say it is Δ_n. If T is a τ-theory and a τ-formula is T-equivalent to both a Σ_n formula and a Π_n formula, we say it is $\Delta_n(T)$.*

Definition 2.3.3. *We say a formula is existential if it is Σ_1, universal if Π_1. We say a theory T is universal if there exists a set of universal sentences $A \subseteq T$ such that $A \models T$. We also say T is axiomatized by its universal part in this case.*
Similarly we can define existential theories.

Remark 2.3.1. *Note that this definition is different from Lévy Hierarchy in set theory. In set theory we allow bounded quantifiers to appear in Σ_0-formulas. We'll define Lévy Hierarchy later.*

Definition 2.3.4. *We also use existential formula to refer to Σ_1 formula, universal formula to refer to Π_1 formula.*

We define a weaker version of $\mathfrak{M} \prec \mathfrak{N}$ here. $\mathfrak{M} \prec \mathfrak{N}$ says everything that exists in the larger structure $\mathfrak{N}$ exists in $\mathfrak{M}$. This condition is sometimes too strong. We define $\mathfrak{M} \prec_1 \mathfrak{N}$ to be everything defined by quantifier-free formula with parameters that exists in the larger structure $\mathfrak{N}$ exists in $\mathfrak{M}$.

Definition 2.3.5. *Let $\mathfrak{M}, \mathfrak{N}$ be τ-structures, we say $\mathfrak{M}$ is existentially closed in $\mathfrak{N}$, denoted by*

$$\mathfrak{M} \prec_1 \mathfrak{N}$$

if $\mathfrak{M} \sqsubseteq \mathfrak{N}$, and for any τ-formula $\phi(\vec{x})$ which is Σ_1, any $\vec{a} \in \mathfrak{M}^k$,

$$\mathfrak{N} \models \phi(\vec{a}) \Rightarrow \mathfrak{M} \models \phi(\vec{a}).$$

We say

$$\mathfrak{M} \equiv_1 \mathfrak{N}$$

if for any τ-sentence σ which is Σ_1,

$$\mathfrak{M} \models \sigma \Leftrightarrow \mathfrak{N} \models \sigma.$$

Remark 2.3.2. *We have a standard model theoretic fact that if* $\mathfrak{M} \sqsubseteq \mathfrak{N}$*, and for any* Σ_1 *formula* $\phi(\vec{x})$*, any* $\vec{a} \in \mathfrak{M}^k$*, if* $\mathfrak{M} \models \phi(\vec{a})$*, then* $\mathfrak{N} \models \phi(\vec{a})$*. Thus, the* $\Rightarrow$ *in the preceding definition for* $\mathfrak{M} \prec_1 \mathfrak{N}$ *can be replaced by* $\Leftrightarrow$*.*

We prove a lemma saying $\prec_1$ preserves Π_2 formulas from a larger structure to a smaller one.

Lemma 2.3.1. *If* $\psi(x)$ *is a* Π_2 *formula,* $\mathfrak{M} \prec_1 \mathfrak{N}$*,* $\vec{a} \in \mathfrak{M}^k$ *and* $\mathfrak{N} \models \psi(\vec{a})$*, then* $\mathfrak{M} \models \psi(\vec{a})$*.*

Proof. $\psi(\vec{x})$ is logically equivalent to $\forall \vec{y} \exists \vec{z} \varphi(\vec{x}, \vec{y}, \vec{z})$, where φ is quantifier-free. For any $\vec{b} \in \mathfrak{M}^l$, since $\mathfrak{M} \prec_1 \mathfrak{N}$, we have

$$\mathfrak{M} \models \exists \vec{z} \varphi(\vec{a}, \vec{b}, \vec{z}) \text{ if and only if } \mathfrak{N} \models \exists \vec{z} \varphi(\vec{a}, \vec{b}, \vec{z}).$$

Thus, $\mathfrak{M} \models \psi(\vec{a})$. □

We now define some notions to single out different parts of the theory in terms of the complexity of sentences.

Definition 2.3.6. *For a set of sentences* T*, we define* $T_\forall, T_{\forall\vee\exists}, T_{\Pi_2}$ *as follows:*
$T_\forall = Cn(\{\sigma : \sigma \text{ is a } \Pi_1 \text{ sentence and } T \models \sigma\})$,
$T_\exists = Cn(\{\sigma : \sigma \text{ is a } \Sigma_1 \text{ sentence and } T \models \sigma\})$,
$T_{\forall\vee\exists} = Cn(T_\forall \cup T_\exists)$,
$T_{\Pi_2} = Cn(\{\sigma : \sigma \text{ is a } \Pi_2 \text{ sentence and } T \models \sigma\})$.
We call $T_\forall$ *the universal part of* T*,* $T_\exists$ *the existential part of* T*,* $T_{\forall\vee\exists}$ *the universal and existential part of* T*, and* T_{Π_2} *the* Π_2 *part of* T*.*

As we just see in Lemma 2.3.1, $\prec_1$ preserves the Π_2 part of the theory of the larger structure into the smaller structure. We now prove more properties involving the universal part of theory and the universal and existential part of the theory.

Proposition 2.3.1. *If* $\mathfrak{M} \models T_\forall$*, there exists* $\mathfrak{N} \models T$ *such that* $\mathfrak{M} \sqsubseteq \mathfrak{N}$*.*

Proof. We only need to prove $Diag(\mathfrak{M}) \cup T$ is consistent. Suppose not, there exists $\psi(\vec{a}) \in Diag(\mathfrak{M})$ such that $T \vdash \neg\psi(\vec{a})$, thus $T \vdash \forall \vec{x} \neg\psi(\vec{x})$. $(\forall \vec{x} \neg\psi(\vec{x})) \in T_\forall$ thus $\mathfrak{M} \models \forall \vec{x} \neg\psi(\vec{x})$, contradiction. □

Proposition 2.3.2. *If* $\mathfrak{M} \models T_{\forall\vee\exists}$*, there exists* $\mathfrak{N} \models T$ *such that* $\mathfrak{M} \sqsubseteq \mathfrak{N}$ *and* $\mathfrak{M} \equiv_1 \mathfrak{N}$*.*

Proof. Suppose not, for any model $\mathfrak{N} \models T$ such that $\mathfrak{M} \sqsubseteq \mathfrak{N}$, let $\sigma_\mathfrak{N}$ be a universal sentence such that $\mathfrak{M} \models \sigma_\mathfrak{N}$ and $\mathfrak{N} \models \neg\sigma_\mathfrak{N}$.

$$T \cap Diag(\mathfrak{M}) \cup \{\sigma_\mathfrak{N} : \mathfrak{M} \sqsubseteq \mathfrak{N} \models T\} \text{ is inconsistent}$$

Otherwise, there exists structure $\mathfrak{U} \models T \cup \{\sigma_\mathfrak{N} : \mathfrak{M} \sqsubseteq \mathfrak{N} \models T\}$ such that $\mathfrak{M} \sqsubseteq \mathfrak{U}$, but $\mathfrak{U} \models \neg\sigma_\mathfrak{U}$, contradiction.
Thus, there exists $\phi(\vec{a}) \in Diag(\mathfrak{M})$ and $\sigma_1, ..., \sigma_n \in \{\sigma_\mathfrak{N} : \mathfrak{M} \sqsubseteq \mathfrak{N} \models T\}$ such that $T \vdash \forall \vec{x} \neg\phi(\vec{x}) \vee \neg \bigwedge_i \sigma_i$. Hence $(\forall \vec{x} \neg\phi(\vec{x}) \vee \neg \bigwedge_i \sigma_i) \in T_{\forall\vee\exists}$, $\mathfrak{M} \models \forall \vec{x} \neg\phi(\vec{x}) \vee \neg \bigwedge_i \sigma_i$, contradiction. □

Model completeness is a good property for a theory such that the substructure relation for any two models of the theory is elementary substructure relation. We define it below.

Definition 2.3.7. *We define a theory S to be model complete if for any models $\mathfrak{M}, \mathfrak{N}$ of S with $\mathfrak{M} \sqsubseteq \mathfrak{N}$, we have $\mathfrak{M} \prec \mathfrak{N}$.*

Definition 2.3.8. *We define theory S to be the model companion of T if S is model complete and $S_\forall = T_\forall$.*

Remark 2.3.3. *We'll show later using Robinson's test that if the model companion of a theory exists, it is unique.*
There are theories that don't have model companions. For an example, see [Tent and Ziegler, 2012, p36, Exercise 3.2.1].

We say a structure $\mathfrak{M}$ is existentially closed in a superstructure $\mathfrak{N}$ if $\mathfrak{M} \prec_1 \mathfrak{N}$. We define an extended notion called T-ec below. A structure is T-ec if it is existentially closed in any superstructure satisfying T.

Definition 2.3.9. *We say $\mathfrak{M}$ is T-ec(existentially closed) if*
(i)$\mathfrak{M} \models T_\forall$,
(ii)For any model $\mathfrak{N} \models T$ such that $\mathfrak{M} \sqsubseteq \mathfrak{N}$, we have $\mathfrak{M} \prec_1 \mathfrak{N}$.

We prove below a lemma about T-ec saying that a structure is T-ec if and only if it is $T_\forall$-ec. This means that the property T-ec is determined by the universal part of T.

Lemma 2.3.2. *$\mathfrak{M}$ is T-ec if and only if it is $T_\forall$-ec.*

Proof. It is easy to show $(T_\forall)_\forall = T_\forall$.
If $\mathfrak{M}$ is T-ec, and $\mathfrak{M} \sqsubseteq \mathfrak{N} \models T_\forall$, there exists $\mathfrak{U} \models T$ such that

$$\mathfrak{M} \sqsubseteq \mathfrak{N} \sqsubseteq \mathfrak{U}.$$

Hence $\mathfrak{M} \prec_1 \mathfrak{U}$. For Σ_1 formula $\psi(\vec{x})$, $\vec{a} \in \mathfrak{M}^k$, if $\mathfrak{N} \models \psi(\vec{a})$, then $\mathfrak{U} \models \psi(\vec{a})$, and since $\mathfrak{M} \prec_1 \mathfrak{U}$, $\mathfrak{M} \models \psi(\vec{a})$. Thus $\mathfrak{M} \prec_1 \mathfrak{N}$. $\mathfrak{M}$ is $T_\forall$-ec.
It is easy to show if $\mathfrak{M}$ is $T_\forall$-ec, then it is T-ec. $\square$

There is another lemma showing that T-ec structures are closed under $\prec_1$ downwards.

Lemma 2.3.3. *If $\mathfrak{M}$ is T-ec and $\mathfrak{N} \prec_1 \mathfrak{M}$, then $\mathfrak{N}$ is T-ec.*

Proof. Since $\mathfrak{N} \prec_1 \mathfrak{M}$ and $\mathfrak{M}$ is T-ec, we have $\mathfrak{N} \models T_\forall$.
If $\mathfrak{N} \sqsubseteq \mathfrak{L} \models T$, we claim $T_\forall \cup Diag(\mathfrak{L}) \cup Diag(\mathfrak{M})$ is consistent. Then by Gödel's Completeness Theorem, there is $\mathfrak{U} \models T_\forall$ such that $\mathfrak{L} \sqsubseteq \mathfrak{U}$, $\mathfrak{M} \prec_1 \mathfrak{U}$ ($\mathfrak{M}$ is $T_\forall$-ec). For any Σ_1 formula $\phi(\vec{x})$ and $\vec{a} \in \mathfrak{N}^k$, if $\mathfrak{L} \models \phi(\vec{a})$, then $\mathfrak{U} \models \phi(\vec{a})$, and since $\mathfrak{N} \prec_1 \mathfrak{U}$, $\mathfrak{N} \models \phi(\vec{a})$. Thus $\mathfrak{N} \prec_1 \mathfrak{L}$.

Proof of the claim $T_\forall \cup Diag(\mathfrak{L}) \cup Diag(\mathfrak{M})$ is consistent:
If not, there are $\psi_\mathfrak{L}(\vec{a},\vec{c}) \in Diag(\mathfrak{L})$, $\psi_\mathfrak{M}(\vec{b},\vec{c}) \in Diag(\mathfrak{M})$, $\vec{a} \in (|\mathfrak{L}| \setminus |\mathfrak{N}|)^{<\omega}$, $\vec{b} \in (|\mathfrak{M}| \setminus |\mathfrak{N}|)^{<\omega}$, $\vec{c} \in |\mathfrak{N}|^{<\omega}$ such that

$$T_\forall \vdash \neg\psi_\mathfrak{L}(\vec{a},\vec{c}) \vee \neg\psi_\mathfrak{M}(\vec{b},\vec{c}).$$

Thus,

$$T_\forall \vdash \forall\vec{x}\neg\psi_\mathfrak{L}(\vec{x},\vec{c}) \vee \forall\vec{y}\neg\psi_\mathfrak{M}(\vec{y},\vec{c}).$$

Since $\mathfrak{L} \models T_\forall$, $\mathfrak{L} \models \forall\vec{y}\neg\psi_\mathfrak{M}(\vec{y},\vec{c})$, then $\mathfrak{M} \models \forall\vec{y}\neg\psi_\mathfrak{M}(\vec{y},\vec{c})$, which contradicts $\mathfrak{M} \models \psi_\mathfrak{M}(\vec{b},\vec{c})$. $\square$

Proposition 2.3.3. *(Robinson's test)For a theory T, the followings are equivalent:*
(i)T is model complete;
(ii)every model of T is T-ec;
(iii)every existential formula is T-equivalent to a Π_1 formula;
(iv)every formula is T-equivalent to a Σ_1 formula.

Proof. (i)⇒(ii) by definition.
(iii)⇒(iv) by induction on the complexity of the formula.
(iv)⇒(i) by definition.
(ii)⇒(iii): Any Σ_1 formula is logically equivalent to some formula $\exists\vec{y}\phi(\vec{x},\vec{y})$ where $\phi(\vec{x},\vec{y})$ is quantifier-free, if $T \cup \{\exists\vec{y}\phi(\vec{x},\vec{y})\}$ is inconsistent,

$$T \vdash \exists\vec{y}\phi(\vec{x},\vec{y}) \leftrightarrow \forall z(z \neq z).$$

Else let

$$\Lambda(\vec{x}) = \{\forall\vec{z}\psi(\vec{x},\vec{z}) : \psi \text{ is quantifier-free and } T; \exists\vec{y}\phi(\vec{x},\vec{y}) \vdash \forall\vec{z}\psi(\vec{x},\vec{z})\},$$

add new constant symbols $\vec{c} = (c_1,...,c_n)$ for tuple $\vec{x}$, $T \cup \Lambda(\vec{c})$ is consistent. For any $\mathfrak{M} \models T \cup \Lambda(\vec{c})$ in language $\tau \cup \{c_1,...,c_n\}$, we claim $T \cup Diag(\mathfrak{M}) \cup \{\exists\vec{y}\phi(\vec{c},\vec{y})\}$ is consistent, where $Diag(\mathfrak{M})$ is taken in language $\tau \cup \{c_1,...,c_n\}$.
By compactness, we only need to prove for any quantifier-free formula $\theta(\vec{c},\vec{a}) \in Diag(\mathfrak{M})$, $T \cup \{\theta(\vec{c},\vec{a}), \exists\vec{y}\phi(\vec{c},\vec{y})\}$ is consistent.
If not, $T; \exists\vec{y}\phi(\vec{c},\vec{y}) \vdash \forall\vec{z}\neg\theta(\vec{c},\vec{z})$, thus $\forall\vec{z}\neg\theta(\vec{c},\vec{z}) \in \Lambda(\vec{c})$, $\mathfrak{M} \models \forall\vec{z}\neg\theta(\vec{c},\vec{z})$ and $\mathfrak{M} \models \theta(\vec{c},\vec{a})$, contradiction, claim holds.
Thus, there exists $\mathfrak{N} \sqsupseteq \mathfrak{M}$ such that $\mathfrak{N} \models T \cup \{\exists y\phi(\vec{c},\vec{y})\}$. Since $\mathfrak{M}$ is T-ec, $\mathfrak{M} \prec_1 \mathfrak{N}$ in language τ and $\mathfrak{M} \models \exists\vec{y}\phi(\vec{c},\vec{y})$. Thus $T \cup \Lambda(\vec{c}) \vdash \exists\vec{y}\phi(\vec{c},\vec{y})$.
By compactness, there are finite formulas $\theta_1(\vec{c}),...,\theta_k(\vec{c})$ in $\Lambda(\vec{c})$ such that $T \cup \{\theta_1(\vec{c}) \wedge ... \wedge \theta_k(\vec{c})\} \vdash \phi(\vec{c},\vec{y})$, thus,

$$T \vdash \theta_1(\vec{x}) \wedge ... \wedge \theta_k(\vec{x}) \leftrightarrow \exists\vec{y}\phi(\vec{x},\vec{y}).$$

And the conjunction of Π_1 sentences is still Π_1, since $\forall\vec{x}\rho(\vec{x},\vec{y}) \wedge \forall\vec{x}o(\vec{x},\vec{z}) \leftrightarrow \forall\vec{x}(\rho(\vec{x},\vec{y}) \wedge o(\vec{x},\vec{z}))$. $\square$

Remark 2.3.4. *From Robinson's test we know that a theory is model complete if it admits quantifier elimination up to Σ_1 level.*

Lemma 2.3.4. *Suppose T is model complete, then*
(i)T is axiomatized by the set of Π_2 sentences

$$A = T_\forall \cup \{\chi : \chi \textit{ is } \forall\vec{x}(\exists\vec{y}\psi(\vec{x},\vec{y}) \leftrightarrow \forall\vec{z}\theta(\vec{x},\vec{z})), \psi, \theta \textit{ are quantifier-free and } T \models \chi\},$$

i.e. $Cn(A) = T$.
(ii)$\mathfrak{M}$ is T-ec if and only if $\mathfrak{M} \models T$.

Proof. (i)Obviously $Cn(A) \subseteq T$. Since $Cn(A)$ satisfies Robinson's test Proposition 2.3.3(iii), $Cn(A)$ is model complete. For any $\mathfrak{M} \models Cn(A)$, by Proposition 2.3.1, there exists $\mathfrak{N} \models T$ such that $\mathfrak{M} \sqsubseteq \mathfrak{N}$, thus $\mathfrak{M} \prec \mathfrak{N}$ and $\mathfrak{M} \models T$. Thus, $Cn(A) \supseteq T$.
(ii)It is easy to show if $\mathfrak{M} \models T$, then $\mathfrak{M}$ is T-ec.
If $\mathfrak{M}$ is T-ec, then $\mathfrak{M} \models T_\forall$. Thus, there exists $\mathfrak{N} \models T$ such that $\mathfrak{M} \prec_1 \mathfrak{N}$. By Lemma 2.3.1, $\mathfrak{M} \models T_{\Pi_2}$. By (i), $T = T_{\Pi_2}$, thus $\mathfrak{M} \models T$. ☐

Lemma 2.3.5. *If theory S is the model companion of theory T,*

$$\mathfrak{M} \models S \Leftrightarrow \mathfrak{M} \textit{ is } T\textit{-ec}$$

Proof. $\mathfrak{M} \models S \Leftrightarrow \mathfrak{M}$ is S-ec $\Leftrightarrow \mathfrak{M}$ is $S_\forall$-ec $\Leftrightarrow \mathfrak{M}$ is $T_\forall$-ec $\Leftrightarrow \mathfrak{M}$ is T-ec.
The first $\Leftrightarrow$ comes from S model complete and Lemma 2.3.4. The second and fourth $\Leftrightarrow$ come from Lemma 2.3.2. The third $\Leftrightarrow$ comes from $S_\forall = T_\forall$. ☐

From Lemma 2.3.5, we know that if there exists a model companion of T, it is the theory of T-ec structures. Thus, it is unique. From Lemma 2.3.4, we also know that the model companion is axiomatized by its Π_2 part.

Motivated by this, we define the Kaiser hull of a theory T to be the Π_2 part of the theory of T-ec structures.

Definition 2.3.10. *We define the Kaiser hull $KH(T)$ of theory T as follows:*

$$KH(T) = Th(\{\mathfrak{M} : \mathfrak{M} \textit{ is } T\textit{-ec}\})_{\Pi_2}.$$

Remark 2.3.5. *From Lemma 2.3.2, we know $KH(T) = KH(T_\forall)$.*

Lemma 2.3.6. *If S is the model companion of T, then $S = KH(T)$. Thus, if the model companion of a theory exists, it is unique.*

Proof. From Lemma 2.3.5, $S = Th\{\mathfrak{M} : \mathfrak{M} \models S\} = Th\{\mathfrak{M} : \mathfrak{M}$ is T-ec$\}$. Since S is model complete, it is axiomatized by Π_2 sentences, and $S = Th\{\mathfrak{M} : \mathfrak{M}$ is T-ec$\}$. ☐

Corollary 2.3.1. *If S is model complete, then $S = KH(S)$.*

Proof. This directly comes from Lemma 2.3.6, since S is the model companion of itself. $\square$

We now define some constraint on sentences that is stronger than being satisfied by all T-ec structures.

Definition 2.3.11. *Let T be a τ-theory. We say a τ-sentence σ is strongly $T_{\forall\vee\exists}$-consistent if for any theory R s.t. $R \supseteq T$, $\sigma + R_{\forall\vee\exists}$ is consistent.*

For a Π_2 sentence, being $T_{\forall\vee\exists}$-consistent implies being in the theory of T-ec structures. We need the following fact to prove this.

Fact 2.3.1. *If T is a complete theory, $T_{\forall\vee\exists} + \sigma$ is consistent, then*

$$(T_{\forall\vee\exists} + \sigma)_{\forall} = T_{\forall}.$$

Proof. Let $A = \{\varrho : \varrho \text{ is a } \Pi_1 \text{ sentence and } T_{\forall\vee\exists} + \sigma \models \varrho\}$ and $B = \{\varrho : \varrho \text{ is a } \Pi_1 \text{ sentence and } T \models \varrho\}$. We only need to prove $A \models B$ and $B \models A$.
For a Π_1 sentence ϱ such that $T \models \varrho$, $\varrho \in T_{\forall\vee\exists}$, hence $A \models \varrho$.
For a Π_1 sentence ϱ such that $T_{\forall\vee\exists} + \sigma \models \varrho$, since T is a complete theory, $T \models \varrho$ or $T \models \neg\varrho$. If $T \models \neg\varrho$, then $\varrho \in T_{\forall\vee\exists}$, contradiction. Thus, $T \models \varrho$, and $B \models \varrho$. $\square$

Lemma 2.3.7. *If a Π_2 sentence σ is strongly $T_{\forall\vee\exists}$-consistent, then $\sigma \in KH(T)$.*

Proof. For any T-ec structure $\mathfrak{M}$, $\mathfrak{M} \models T_{\forall}$. By Proposition 2.3.1, there exists $\mathfrak{N} \models T$ such that

$$\mathfrak{M} \sqsubseteq \mathfrak{N}.$$

For any Π_2 sentence σ that is $T_{\forall\vee\exists}$-consistent, $(Th(\mathfrak{N})_{\forall\vee\exists} + \sigma)$ is consistent. Since $Th(\mathfrak{N})$ is complete,

$$(Th(\mathfrak{N})_{\forall\vee\exists} + \sigma)_{\forall} = Th(\mathfrak{N})_{\forall}.$$

By Proposition 2.3.1, there exists structure $\mathfrak{U} \models Th(\mathfrak{N})_{\forall\vee\exists} + \sigma$ such that

$$\mathfrak{M} \sqsubseteq \mathfrak{N} \sqsubseteq \mathfrak{U}.$$

By Lemma 2.3.2, $\mathfrak{M}$ is $T_{\forall}$-ec. we have $\mathfrak{U} \models T_{\forall}$, thus $\mathfrak{M} \prec_1 \mathfrak{U}$. By Lemma 2.3.1, $\mathfrak{M} \models \sigma$.

$\square$

By imposing an additional condition on the model companion S of theory T, we can prove that model companion is stable under extensions of the theory. This means that with this additional condition, from S we can construct the model companion of any theory extending T in a reasonable way. We prove this Lemma below.

Lemma 2.3.8. *If S is the model companion of T, and $S_{\forall\vee\exists} = T_{\forall\vee\exists}$, then for any theory $R \supseteq T$, $R_{\forall\vee\exists} + S$ is the model companion of R.*

Proof. Since S is model complete, $R_{\forall\vee\exists}+S$ is model complete.
Obviously, $R_{\forall\vee\exists} \subseteq (R_{\forall\vee\exists}+S)_{\forall\vee\exists}$. We want to show $R_{\forall\vee\exists} \supseteq (R_{\forall\vee\exists}+S)_{\forall\vee\exists}$.
For any

$$\mathfrak{M} \models R,$$

since $R \supseteq T \supseteq T_{\forall\vee\exists} = S_{\forall\vee\exists}$, by Proposition 2.3.2, there exists $\mathfrak{N} \models S$ such that

$$\mathfrak{M} \equiv_1 \mathfrak{N}.$$

Then $\mathfrak{N} \models R_{\forall\vee\exists}+S$, therefore,

$$\mathfrak{M} \models (R_{\forall\vee\exists}+S)_{\forall\vee\exists}.$$

Thus, $R_{\forall\vee\exists} \supseteq (R_{\forall\vee\exists}+S)_{\forall\vee\exists}$. □

Definition 2.3.12. *Given a chain of τ-structures*

$$\mathfrak{M}_0 \sqsubseteq \mathfrak{M}_1 \sqsubseteq \mathfrak{M}_2 \sqsubseteq ...,$$

we define $\bigcup_n \mathfrak{M}_n$ to be τ-structure with domain $|\mathfrak{M}| = \cup_i |\mathfrak{M}_n|$, relations $R^{\mathfrak{M}} = \cup_n R^{\mathfrak{M}_n}$, functions $f^{\mathfrak{M}} = \cup_n f^{\mathfrak{M}_n}$, and constants $c^{\mathfrak{M}} = c^{\mathfrak{M}_0}$.

We state without proving the following lemma. The proof is a direct check of definitions.

Lemma 2.3.9. *For an infinite chain of structures*

$$\mathfrak{M}_0 \sqsubseteq \mathfrak{M}_1 \sqsubseteq \mathfrak{M}_2 \sqsubseteq ...,$$

and a Π_2-sentence σ, if there are infinite many structures in this chain satisfying this sentence, then $\bigcup_n \mathfrak{M}_n$ satisfies this sentence.

Then we prove the following theorem that will be useful when studying CH in model companion of set theory.

Proposition 2.3.4. *If two sets S, T of sentences are consistent, axiomatized by Π_2 sentences, and $S_{\forall\vee\exists}+T_{\forall\vee\exists}$ is consistent, then $S+T$ is consistent.*

Proof. Since $S_{\forall\vee\exists}+T_{\forall\vee\exists}$ is consistent, there is a model $\mathfrak{M}_0 \models S_{\forall\vee\exists}+T_{\forall\vee\exists}$. By Proposition 2.3.2, there exists $\mathfrak{M}_1 \models S$ such that $\mathfrak{M}_0 \sqsubseteq \mathfrak{M}_1$ and $\mathfrak{M}_0 \equiv_1 \mathfrak{M}_1$.
Then $\mathfrak{M}_1 \models S_{\forall\vee\exists}+T_{\forall\vee\exists}$, since it has the same universal and existential parts as $\mathfrak{M}_0$.
We apply Proposition 2.3.2 again and get $\mathfrak{M}_2 \models T$ such that $\mathfrak{M}_1 \sqsubseteq \mathfrak{M}_2$ and $\mathfrak{M} \equiv_1 \mathfrak{M}_2$.
By recursion, we get a chain of structures

$$\mathfrak{M}_0 \sqsubseteq \mathfrak{M}_1 \sqsubseteq \mathfrak{M}_2 \sqsubseteq ...,$$

where $\mathfrak{M}_n \models S_{\forall\vee\exists}+T_{\forall\vee\exists}$, $\mathfrak{M}_{2n+1} \models S$, $\mathfrak{M}_{2n+2} \models T$.
Since S, T are both axiomatized by Π_2 sentences, from Lemma 2.3.9 we have $\bigcup_n \mathfrak{M}_n \models S+T$. □

2.4 Partial Morleyizations

This section proceeds with a model theoretic point of view. We first display how to add symbols to represent definable relations and functions.

Definition 2.4.1. *Given a τ-formula $\phi(\vec{x})$, define Ax^0_ϕ to be $\forall\vec{x}(\phi(\vec{x}) \leftrightarrow R_\phi(\vec{x}))$ in language $\tau \cup \{R_\phi\}$.*
Given a τ-formula $\psi(\vec{x}, y)$, define Ax^1_ψ to be $\forall\vec{x}[(\exists! y\psi(\vec{x}, y) \wedge f_\psi(\vec{x}) = y) \vee (\neg\exists! y\psi(\vec{x}, y) \wedge f_\psi(\vec{x}) = c)]$ in language $\tau \cup \{f_\psi, c\}$.

Remark 2.4.1. *If there is a constant symbol in τ, we can take c to be the original constant symbol in τ. If formula ψ only has one free variable y, it defines a constant symbol f_ψ.*

Definition 2.4.2. *For $A \subseteq \{0,1\} \times \{\phi : \phi \text{ is } \tau\text{-formula}\}$, we can define τ_A to be $\tau \cup \{R_\phi : \langle 0, \phi\rangle \in A\} \cup \{f_\psi : \langle 1, \psi\rangle \in A\}$, and P_A to be $\{Ax^0_\phi : \langle 0, \phi\rangle \in A\} \cup \{Ax^1_\psi : \langle 1, \psi\rangle \in A\}$.*

Fact 2.4.1. *For any model $\mathfrak{M}$ in language τ, any $A \subseteq \{0,1\} \times \{\phi : \phi \text{ is } \tau\text{-formula}\}$, there is a unique extension $\mathfrak{N}$ in language τ_A such that $\mathfrak{N}$ restricted to τ is $\mathfrak{M}$ and $\mathfrak{N} \models P_A$.*

Now we partially Morleyize the language of set theory $\{\in\}$. The reason to do this is that in set theory, formulas with bounded quantifiers are absolute for transitive models. However, in model theory, only formulas with no quantifiers are Δ_0. We need to add relation symbols to make formulas in $\{\in\}$ with bounded quantifiers equivalent to atomic formulas in the extended language.

We define Lévy Hierarchy here.

Definition 2.4.3. *(Lévy Hierarchy) In the language of set theory, given a theory T, we say*
(i)A formula is $\Sigma_0(T)$ in set theory if it is T-equivalent to a formula with only bounded quantifier.
(ii)A formula is $\Pi_n(T)$ in set theory if it is T-equivalent to a formula of form $\neg\psi(\vec{x})$ where ψ is $\Sigma_n(T)$.
(iii)A formula is $\Sigma_{n+1}(T)$ in set theory if it is T-equivalent to a formula of form $\exists\vec{x}\psi(\vec{x}, \vec{y})$, where ψ is $\Pi_n(T)$.
(iv)A formula is $\Delta_n(T)$ in set theory if it is both $\Sigma_n(T)$ and $\Pi_n(T)$ in set theory.

Remark 2.4.2. *We'll always add "in set theory" after the complexity if we refer to Lévy Hierarchy, to distinguish from the complexity defined in model theory.*

Remark 2.4.3. *When we are working in $T = ZF^-, ZF, ZFC$, etc, or $T = \emptyset$, we'll say Σ_n, Π_n, Δ_n in set theory instead of $\Sigma_n(T)$, $\Pi_n(T)$, $\Delta_n(T)$ in set theory for simplicity.*

Now we can partially Morleyize the language of set theory to make Δ_0 formulas in set theory atomic.

Definition 2.4.4. *Let* $A_{\Delta_0} = \{\langle 0, \phi\rangle : \phi$ *is a* Δ_0 *formula in set theory*$\} \cup$ $\{\langle 1, \psi\rangle : \psi$ *is a* Δ_0*-formula in set theory defining the graph of a Gödel operation*$\} \cup$ $\{\langle 1, \psi_\emptyset\rangle\} \cup \{\langle 1, \psi_\omega\rangle\}$*, where*

$$\psi_\emptyset \equiv \forall x \in \emptyset(\neg x = x),$$

$$\psi_\omega \equiv \forall x \in \omega[(x = \emptyset \vee \exists y \in \omega(x = y + 1)) \wedge (x + 1 \in \omega)].$$

We define the language $\in_{\Delta_0}$ *to be*

$$\in_{\Delta_0} = \{\in\}_{A_{\Delta_0}} = \{\in\} \cup \{R_\phi : \langle 0, \phi\rangle \in A\} \cup \{f_\psi : \langle 1, \psi\rangle \in A\} \cup \{\emptyset, \omega\}.$$

Accordingly, we need to modify ZFC to suit this new language $\in_{\Delta_0}$.

Definition 2.4.5. *Define theory* $Z^-_{\Delta_0}$ *in the language* $\in_{\Delta_0}$ *to be*
$Z^-_{\Delta_0}$ = *Extensionality, Foundation,* $\psi_\emptyset$, ψ_ω,

$$Ax^0_\phi \text{ for every quantifier-free formula } \phi \text{ in set theory,}$$

for every Δ_0 *formula* ϕ *in set theory,*

$$\forall \vec{x}(R_{\neg\phi}(\vec{x}) \leftrightarrow \neg R_\phi(\vec{x})),$$

$$\forall \vec{x}[R_{\phi\vee\psi}(\vec{x}) \leftrightarrow (R_\phi(\vec{x}) \vee R_\psi(\vec{x}))],$$

$$\forall \vec{x}(R_{\exists y\in z\phi}(\vec{x}, z) \leftrightarrow \exists y \in z R_\phi(\vec{x}, y)),$$

for $\langle 1, \psi\rangle \in A_{\Delta_0}$,

$$\forall \vec{x}, y(R_\psi(\vec{x}, y) \leftrightarrow f_\psi(\vec{x}) = y).$$

Remark 2.4.4. *We take the pain to define* $Z^-_{\Delta_0}$ *this way to make it universal in* $\in_{\Delta_0}$. *We can show that* $Z^-_{\Delta_0}$ *is logically equivalent to Extensionality+Foundation+*$P_{A_{\Delta_0}}$.

Definition 2.4.6. $ZF^-_{\Delta_0}$ *is* $Z^-_{\Delta_0}$+ *Replacement for all* $\in_{\Delta_0}$*-formulas,*
$ZFC^-_{\Delta_0}$ *is* $ZF^-_{\Delta_0} + AC$,
ZFC_{Δ_0} *is* $ZFC^-_{\Delta_0}$+ *Power Set.*

Remark 2.4.5. $Z^-_{\Delta_0} \vdash$ *Pairing* $\wedge$ *Union. This is because both Pairing and Union are Gödel operations and* $\forall \vec{x}, y(R_\phi(\vec{x}, y) \leftrightarrow f_\phi(\vec{x}) = y)$ *for* ϕ Δ_0*-formula which is the graph of a Gödel operation.*
$ZF^-_{\Delta_0} \vdash$ *Separation for all* $\in_{\Delta_0}$*-formulas. This is because we have Replacement and* $\forall X, a, p\exists Y\ Y = \{y : u \in X \wedge [(\psi(u, p) \wedge y = u) \vee (\neg\psi(u, p) \wedge y = a)]\} \vdash$ $\forall X, p\exists Y(\neg\exists u\psi(u, p) \to Y = \emptyset) \vee (\exists u\psi(u, p) \to Y = \{u \in X : \psi(u, p)\})$.

Remark 2.4.6. *It is easy to check that Lévy Hierarchy assuming* ZF^- *and the complexity of formula in language* $\in_{\Delta_0}$ *assuming* $ZF^-_{\Delta_0}$ *coincides on* Δ_1 *and higher level.*

$\in_{\Delta_0}$ is the minimal language we have to take when dealing with set theory from a model theoretic point of view, since we need the atomic formulas in this language to at least contain all Δ_0 formulas in Lévy Hierarchy. However, this is not enough if we want to consider the model companion of set theory. As we will show afterwards, there is no model companion of set theory in language $\in_{\Delta_0}$. Thus, we need to consider extended languages of $\in_{\Delta_0}$.

Definition 2.4.7. *For any A such that $A_{\Delta_0} \subseteq A \subseteq \{0,1\}\times\{\phi : \phi$ is a formula in set theory$\}$, let $\tau = \{\in\}_A$, we say τ is a definable language (defined by A) and we define*
Z_τ^- is $Z_{\Delta_0} + P_A$,
ZF_τ^- is Z_τ^-+ Replacement for all $\in_{\Delta_0}$-formulas,
ZFC_τ^- is $ZF_\tau^- + AC$,
ZFC_τ is ZFC_τ^-+ Power Set.

Remark 2.4.7. *Since this language is definable, $ZF_\tau^- \vdash$ Relpacement for all τ-formulas.*
The same as $\in_{\Delta_0}$, we have $ZF_\tau^- \vdash$ Pairing $\wedge$ Union $\wedge$ Separation for all τ-formulas.

We also need to consider partial Morleyizations in different structures.

Definition 2.4.8. *Given a set M, $A \subseteq \{0,1\}\times\{\phi : \phi$ is a formula in set theory$\}$, let $\tau = \{\in\}_A$, from Fact 2.4.1 there exists a unique extension $\mathfrak{N}$ of $(M, \in)$ in language τ such that $\mathfrak{N} \models P_A$. We denote $\mathfrak{N}$ as (M, τ^M).*
For a proper class M, we can also have (M, τ^M), where function, relation, constant symbols in this language are defined by formulas in set theory relativized to M.

3 Absoluteness Theorems

3.1 Mostowski's Absoluteness Theorem

This section follows from [Jech, 2006].

Definition 3.1.1. *We say T is a tree on K if $T \subseteq Seq(K)$ and $\forall s \in T \forall n < dom(s)$ $(s \upharpoonright n \in T)$.*
We define the order on tree T to be $s \geq t$ iff $s \subseteq t$.
We say $x \in K^\omega$ is a path in T if $\forall n \in \omega$ $x \upharpoonright n \in T$, and

$$[T] = \{x : x \text{ is a path in } T\}.$$

Remark 3.1.1. *Afterwards, we wouldn't distinguish $h \in Seq(K^2)$ and $\langle s, t\rangle \in Seq(K)^2$ if $dom(h) = dom(s) = dom(t) \wedge \forall n < dom(h)(\langle s(n), t(n)\rangle = h(n))$. Thus, we also view $T \subseteq Seq(K)^2$ as a tree if*

$$\forall \langle s, t\rangle \in T(dom(s) = dom(t) \wedge \forall n < dom(s)(\langle s \upharpoonright n, t \upharpoonright n\rangle \in T)).$$

Lemma 3.1.1. *A is Σ^1_1 if and only if there exists a Δ^0_1 tree T s.t.*

$$x \in A \leftrightarrow T(x) \textit{ is ill-founded.}$$

Proof. If A is Σ^1_1, there is a quantifier-free φ,

$$x \in A \Leftrightarrow \exists y \in \omega^\omega \forall n \in \omega \varphi(x \upharpoonright n, y \upharpoonright n).$$

Let $T = \{(s,t) : dom(s) = dom(t) \wedge \forall n \leq dom(s) \varphi(s \upharpoonright n, t \upharpoonright n)\}$, then T is a Δ^0_1 tree,

$$x \in A \Leftrightarrow \exists y \in \omega^\omega \forall n \in \omega (x \upharpoonright n, y \upharpoonright n) \in T \Leftrightarrow T(x) \text{ is ill-founded.}$$

The other direction is obvious. □

Remark 3.1.2. *The preceding lemma can be relativized to $a \in \omega^\omega$.*

Definition 3.1.2. *We say a transitive model M is adequate if M satisfies enough of ZF^- so that $\omega^M = \omega$, $Seq^M = Seq$, and $M \models$ any well-founded tree has a rank function.*

Theorem 3.1.1. *(Mostowski's Absoluteness Theorem) Let A be $\Sigma^1_1(a)$, M be a transitive adequate model such that $a \in M$, then A is absolute for M.*

Proof. For any $\Delta^0_1(a)$ tree T, $T^M = T$ and for any $x \in \omega^\omega \cap M$, $T(x)^M = T(x)$. If A is $\Sigma^1_1(a)$, there is a $\Delta^0_1(a)$ tree T, $x \in A \leftrightarrow T(x)$ is ill-founded. We only need to show "well-foundedness" is absolute for transitive models.
Let $T_0 \in M$ be a tree on ω, we claim

$$M \models T_0 \text{ is well-founded} \Leftrightarrow V \models T_0 \text{ is well-founded.}$$

If $M \models T_0$ is ill-founded, then $M \models \exists y \in \omega^\omega (y \in [T_0])$, thus $V \models T_0$ is ill-founded.
If $M \models T_0$ is well-founded, then $M \models \exists f : T(x) \to Ord$ which preserves order, thus $V \models T_0$ is well-founded. □

Definition 3.1.3. *We say A is $\mathbf{\Sigma}_n$ in $(H_{\omega_1}, \in)$ if there is a Σ_n formula $\vartheta(x, \vec{y})$ in set theory and $\vec{a} \in |H_{\omega_1}|^k$ such that*

$$x \in A \leftrightarrow (H_{\omega_1}, \in) \models \vartheta(x, \vec{a}).$$

Theorem 3.1.2. *For $n \geq 1$, $A \subseteq \omega^\omega$ is $\mathbf{\Sigma}^1_{n+1}$ if and only if it is $\mathbf{\Sigma}_n$ in $(H_{\omega_1}, \in)$.*

Proof. By Mostowski's Absoluteness Theorem, if $A \subseteq \omega^\omega$ is $\mathbf{\Pi}^1_1$ and defined by $\phi(x, \vec{a})$, where $\vec{a} \in (\omega^\omega)^k$,

$$x \in A \Leftrightarrow (H_{\omega_1}, \in) \models \exists M (M \text{ is transitive and adequate for } A \wedge M \models \phi(x, \vec{a})),$$

Thus, A is $\mathbf{\Sigma}_1$ in $(H_{\omega_1}, \in)$.
If A is $\mathbf{\Sigma}^1_2$, there is a $\mathbf{\Pi}^1_1$ set $B \subseteq (\omega^\omega)^2$ such that

$$A(x) \Leftrightarrow \exists y \in \omega^\omega B(x, y).$$

Then A is $\mathbf{\Sigma}_1$ in H_{ω_1}.
By induction if A is $\mathbf{\Sigma}^1_{n+1}$, A is $\mathbf{\Sigma}_n$ in $(H_{\omega_1}, \in)$.
If $A \subseteq \omega^\omega$ is $\mathbf{\Sigma}_n$ in $(H_{\omega_1}, \in)$, i.e.

$$x \in A \Leftrightarrow (H_{\omega_1}, \in) \models \exists x_1 ... Q_n x_n \phi(x, x_1, ..., x_n, \vec{a}),$$

where $\vec{a} \in (H_{\omega_1})^k$, ϕ is quantifier-free and Q_n is either $\forall$ or $\exists$ depending on n.
By Theorem 2.2.1,

$$x \in A \Leftrightarrow (WF/\sim, E/\sim) \models \exists X \exists X_1 \forall ... Q_n X_n [\phi(X, X_1, ..., X_n, \vec{A}) \wedge Code([X]_\sim) = x],$$

where $Code([\vec{A}]_\sim) = \vec{a}$.
We do not have a simple way to describe Mostowski's collapse in arithmetic, so instead we calculate a particular X out to code x. Let

$$f : trcl(\{x\}) \to \omega,$$

where $f(x) = 0$, $f(\langle n, m\rangle) = 3n + 1$, $f(\{n, m\}) = 3n + 2$, $f(n) = 3n + 3$. Then let $R_x \subseteq \omega \times \omega$ be induced by f so that

$$(trcl(\{x\}), \in) \cong (\omega, R_x).$$

Let $Code([X]_\sim) = x$ be replaced by $R_x = X$, this is recursive.
Thus if A is $\mathbf{\Sigma}_n$ in $(H_{\omega_1}, \in)$, it is $\mathbf{\Sigma}^1_{n+1}$. □

Remark 3.1.3. *This gives directly that $B \subseteq \omega^\omega$ is $\mathbf{\Pi}^1_{n+1}$ if and only if it is $\mathbf{\Pi}_n$ in $(H_{\omega_1}, \in)$. This is because $x \in \omega^\omega$ is Δ_1-definable in $(H_{\omega_1}, \in)$.*
$B \subseteq \omega^\omega$ is $\mathbf{\Pi}^1_{n+1}$ if and only if $\omega^\omega \setminus B$ is $\mathbf{\Sigma}^1_{n+1}$ if and only if $\omega^\omega \setminus B$ is $\mathbf{\Sigma}_n$ in $(H_{\omega_1}, \in)$ if and only if $\omega^\omega \setminus (\omega^\omega \setminus B) = B$ is $\mathbf{\Pi}_n$ in $(H_{\omega_1}, \in)$.

Remark 3.1.4. *Using the same proof, we can get the lightface case:*
For $n \geq 1$, $A \subseteq \omega^\omega$ is Σ^1_{n+1} if and only if it is Σ_n in $(H_{\omega_1}, \in)$.

3.2 Shoenfield's Absoluteness Theorem

This section follows from [Jech, 2006].

Theorem 3.2.1. *(Shoenfield's Absoluteness Theorem)(ZF) Let A be $\Sigma^1_2(a)$, M be a transitive adequate model such that $a \in M$, $\omega_1 \in M$, then A is absolute for M.*

Proof. Since $A = \{x : A(x)\}$ be $\Sigma^1_2(a)$, there is a quantifier-free ϕ_A such that

$$A(x) \Leftrightarrow \exists y \in \omega^\omega \forall z \in \omega^\omega \exists n \in \omega \phi_A(x \restriction n, y \restriction n, z \restriction n, a \restriction n),$$

where ϕ_A is $\Delta^1_1(a)$. Let $T = \{(s, t, r) \in Seq^3 : dom(s) = dom(t) = dom(r) \wedge \forall n \leq dom(s) \neg\phi_A(s \restriction n, t \restriction n, r \restriction n)\}$, $T(x, y) = \{r \in Seq : (x \restriction n, y \restriction n, r) \in T\}$ then

$$A(x) \Leftrightarrow \exists y\ T(x, y) \text{ is well-founded.}$$

It is easy to check that $T^M = T$, if $x, y \in M$, then $T(x,y)^M = T(x,y)$.
We want to show there exists tree U on $\omega\times\omega_1$ such that $x \in A \Leftrightarrow U(x)$ is ill-founded and $U^M = U$.

$$\begin{aligned} A(x) &\Leftrightarrow \exists y\ T(x,y) \text{ is well-founded} \\ &\Leftrightarrow \exists y : \omega \to \omega\ \exists f : Seq \to \omega_1\ f \restriction T(x,y) \text{ is order-preserving} \end{aligned}$$

Fix recursive $\Gamma : \omega \to Seq$ such that Γ is onto and $dom(\Gamma(n)) \leq n$. Let $\Lambda : Seq \to \omega$ to be $\Lambda(s) = min\ \Gamma^{-1}(s)$.
Fix $\tau : \omega \times \omega_1 \to \omega_1$ bijection and $\tau \in M$.
For $t \in Seq(\omega_1)$, let $\langle t_y(k), t_f(k)\rangle = \tau^{-1}(t(k)), \forall k < dom(t)$.
For $s, t \in Seq$ and $dom(s) = dom(t)$, let

$$T(s,t) = \{r \in Seq : (s \restriction dom(r), t \restriction dom(r), r) \in T\}.$$

For $h \in Seq(\omega_1)$, $s, t \in Seq$ and $dom(h) = dom(s) = dom(t)$, let

$$h \restriction T(s,t) = \{\langle w, \alpha\rangle \in Seq \times \omega_1 : w \in T(s,t) \wedge h(\Lambda(w)) = \alpha\}.$$

Let $U = \{(s,t) \in Seq \times Seq(\omega_1) : t_f \restriction T(s, t_y) \text{ is order-preserving}\}$,

$$A(x) \Leftrightarrow \exists g : \omega \to \omega_1 \forall n\ (x \restriction n, g \restriction n) \in U \Leftrightarrow U(x) \text{ is ill-founded.}$$

From the construction, $U^M = U$. For $x \in M$, $U(x)^M = U(x)$. From the proof of Mostowski's Absoluteness theorem, "well-foundedness" is absolute for M, then

$$x \in A \cap M \Leftrightarrow x \in M \wedge U(x) \text{ is ill-founded} \Leftrightarrow M \models U(x) \text{ is ill-founded} \Leftrightarrow M \models A(x).$$

□

Remark 3.2.1. *The same result holds for $\Pi^1_2(a)$ since A is $\Pi^1_2(a)$ iff $\omega^\omega \setminus A$ is $\Sigma^1_2(a)$.*

Remark 3.2.2. *In [Jech, 2006], it is assumed in the condition in Shoenfield's Absoluteness Theorem that $M \models DC$. By carefully checking the proof of Shoenfield's Absoluteness Theorem, we find that DC is not needed in the proof.*

Theorem 3.2.2. *If a sentence σ is $\mathbf{\Sigma}^1_2$ or $\mathbf{\Pi}^1_2$, and $ZFC \vdash \sigma$, then $ZF \vdash \sigma$.*

Proof. Let $a \in \omega^\omega$ be the parameter for σ so that σ is either $\mathbf{\Sigma}^1_2$ or $\mathbf{\Pi}^1_2$. If $V \models ZF$, then $L[a]^V \models ZFC$, thus $L[a]^V \models \sigma$. By Shoenfield's Absoluteness Theorem, $V \models \sigma$. □

This theorem has a significant philosophical implication. It tells us that if a mathematical statement is simple enough, and we proved it using AC, we can actually convert this proof into a proof without using AC.

And it turned out that in most theorems, the statements are simple enough to fall into this category. This is because in mathematics, we usually only study concrete objects and come up with statements that do not require many layers of quantifiers. For example, we study reals, which can be described using elements in ω^ω, or continuous functions from reals to reals, which is characterized by $f(\mathbb{Q})$, thus can also be described using elements in ω^ω. In algebra, we also have a similar case that most theorems are talking about concrete mathematical objects which are not complicated at all. For example, finite fields, finite dimension vector space. In geometry, our focus are usually finite dimensional Euclidean space, which can be viewed as $(\omega^\omega)^n$. In discrete math, we are studying finite sets.

There are theorems that are general enough that it is quantifying over things exceeding ω^ω. In this case we can not apply the above theorem and the use of AC might become necessary. However, we should note that even in this case, when we apply the general form of theorem we proved in mathematics(AC), we usually apply it to concrete objects that can be described using elements in ω^ω.

This implies that you are free to use AC in most mathematical proofs, even if you don't believe in AC. Because every proof using AC can be turned into another proof without using AC, given that your statement is $\mathbf{\Sigma}^1_2$ or $\mathbf{\Pi}^1_2(a)$.

We give an example of statement that satisfies the condition of Theorem 3.2.2, to convince you. And also an example of statement that does not satisfy the condition of Theorem 3.2.2.

Example 3.2.1. *Recall intermediate value theorem in analysis.*
If f is a continuous function from $[a,b] \subseteq \mathbb{R}$ to $\mathbb{R}$, and without lose of generality, suppose $f(a) < f(b)$, then for any $y \in (f(a), f(b))$, there exists $x \in (a,b)$ such that $f(x) = y$.
This theorem is formalized in 2nd-order arithmetic by the following sentence

$$\forall f \in \omega^\omega[(f \text{ codes a continuous function} : \mathbb{Q} \to \mathbb{R}) \to \forall y \in \omega^\omega$$
$$([f(a) < y < f(b)] \to [\exists x \in \omega^\omega(x \in (a,b) \wedge f(x) = y)])].$$

You can check this is $\mathbf{\Pi}^1_2$.

Example 3.2.2. *Let R be a ring. If $I \subseteq R$ is a nontrivial idea, then there is a maximal ideal $M \subseteq R$ such that $I \subseteq M$.*
The proof of this theorem uses Zorn's Lemma. It turns out that the use of Zorn's Lamma is necessary here. The reason why this statement is not $\mathbf{\Pi}^1_2$ or $\mathbf{\Sigma}^1_2$ is that it asks a quantifier over all rings. By Löwenheim Skolem Theorem, we have an infinite ring, thus given any infinite cardinal, we have a ring with this caridnality. Hence, this statement is not expressible in 2nd-order arithmetic.

3.3 Levy's Absoluteness Theorem

We first prove some simple facts about H_{κ^+}.

Proposition 3.3.1. *(ZFC)If κ is an infinite cardinal,*
(i)$(H_{\kappa^+},\in)\models\forall x\exists f(f:\kappa\to x$ is a surjection),
(ii)$(H_{\kappa^+},\in)\models ZFC^-$,
(iii)H_{κ^+} is the unique transitive model of $ZFC^-+\forall x\exists f(f:\kappa\to x$ is a surjection) and contains $\mathcal{P}(\kappa)$.

Proof. It is easy to check (i)(ii), we only prove (iii) here.
If M is a transitive model of $ZFC^-+\forall x\exists f(f:\kappa\to x$ is a surjection) and contains $\mathcal{P}(\kappa)$, $\forall x\in M, trcl(x)^M=trcl(x)$ and $M\models\exists f(f:\kappa\to trcl(x)$ is a surjection), thus $M\subseteq H_{\kappa^+}$.
We claim that $H_{\kappa^+}\cap Ord\subseteq M$. Since $\mathcal{P}(\kappa)\subseteq M$, $\kappa\in M$. If $H_{\kappa^+}\cap Ord$ and $\alpha>\kappa$, $\exists f:\alpha\to\kappa$ bijection, thus there exists $R\subseteq\kappa^2$, $(\alpha,\in)\cong(\kappa,R)$. $\kappa^2\in M$ and there is a bijection $f:\kappa\to\kappa^2$ such that $f\in M$, thus $R\in M$, and Mostowski collapse $\Pi_R^M:(\kappa,R)\to(\alpha,\in)$ has range $\alpha\in M$.
If $H_{\kappa^+}\not\subseteq M$, there exists $x\in H_{\kappa^+}\setminus M$ such that $rank(x)=min\{rank(y):y\in H_{\kappa^+}\setminus M\}$. $x\in H_{\kappa^+}$, thus $\beta=rank(x)\in H_{\kappa^+}$. For all $y\in x$, $rank(y)=rank(y)^M<\beta$, thus $x\subseteq V_\beta^M$. Since $V_\beta^M\in M$, there exists $g:\kappa\to V_\beta^M$ surjection such that $g\in M$. $g^{-1}(x)\subseteq\kappa$ thus $g^{-1}(x)\in M$. $x=g(g^{-1}(x))\in M$, contradiction. ☐

We formulate Levy's Absoluteness Theorem in model theory with a language that extends $\{\in\}$.
We use downward Loewenheim-Skolem theorem to find an elementary substructure and collapse it into a transitive model with small cardinality in the proof.

Definition 3.3.1. *Let $A_{\Delta_1}=\{\langle 0,R_\phi\rangle:\phi$ is $\Delta_1(ZFC^-)\}\cup\{\langle 1,f_\psi\rangle:\psi$ is $\Delta_1(ZFC^-)$ and $ZFC^-\vdash\forall\vec{x}\exists!y\psi(\vec{x},y)\}$. We define language $\in_{\Delta_1}=\in_{\Delta_0}\cup\{R_\phi:\langle 0,R_\phi\rangle\in A_{\Delta_1}\}\cup\{f_\psi:\langle 1,f_\psi\rangle\in A_{\Delta_1}\}$, and denote $ZFC_{\in_{\Delta_1}}$ by ZFC_{Δ_1} for short.*

Theorem 3.3.1. *(Levy's Absoluteness Theorem)(ZFC) Let κ be an infinite cardinal, and sets $A_i\subseteq\mathcal{P}(\kappa)$ for each $i=1,2,...,l$ (hence $A_i\subseteq H_{\kappa^+}$). Then in language $\tau=\in_{\Delta_1}\cup\{A_1,...,A_l\}$, we have*

$$(H_{\kappa^+},\in_{\Delta_1}^{H_{\kappa^+}},A_1,...,A_l)\prec_1(V,\in_{\Delta_1}^V,A_1,...,A_l).$$

This is Lemma 3.1 in [Viale, 2021].

Proof. Since $H_{\kappa^+}\models ZFC^-$ and Δ_1 formulas in set theory are absolute for H_{κ^+},

$$(H_{\kappa^+},\in_{\Delta_1}^{H_{\kappa^+}},A_1,...,A_l)\sqsubseteq(V,\in_{\Delta_1}^V,A_1,...,A_l).$$

If $(V,\in_{\Delta_1}^V,A_1,...,A_l)\models\exists\vec{x}\varphi(\vec{x},\vec{a})$, where $\vec{a}\in(H_{\kappa^+})^m$, we replace every relation symbols and function symbols other then $=,\in,A_1,...,A_l$ in φ with the formulas defining them to get $\exists\vec{z}\phi(\vec{z},\vec{a})$ with ϕ quantifier-free in $\{\in,A_1,...,A_l\}$ such that

$$(H_{\kappa^+},\in_{\Delta_1}^{H_{\kappa^+}},A_1,...,A_l)\vdash\forall\vec{y}(\exists\vec{x}\varphi(\vec{x},\vec{y})\leftrightarrow\exists\vec{z}\phi(\vec{z},\vec{y})),$$

$$(V, \in_{\Delta_1}^V, A_1, ..., A_l) \vdash \forall \vec{y}(\exists \vec{x}\varphi(\vec{x}, \vec{y}) \leftrightarrow \exists \vec{z}\phi(\vec{z}, \vec{y})).$$

There exists $\vec{b} \in V$ such that $(V, \in, A_1, .., A_l) \models \phi(\vec{b}, \vec{a})$.
We take enough large cardinal $\delta \geq \kappa$ such that $\vec{a}, \vec{b} \in H_{\delta^+}$. Then

$$(H_{\delta^+}, \in, A_1, .., A_l) \models \exists \vec{z}\phi(\vec{z}, \vec{a}).$$

By Löewenheim-Skolem theorem there is a structure $(X, E, A_1 \cap X, ..., A_l \cap X)$ such that $|X| = \kappa$, $\kappa \subseteq X$, $trcl(\{a_i\}) \subseteq X$ for each i and

$$(X, E, A_1 \cap X, ..., A_l \cap X) \prec (H_{\delta^+}, \in, A_1, ..., A_l).$$

Take the Mostowski collapse of X:

$$\Pi_X : (X, E, A_1 \cap X, ..., A_l \cap X) \to (N, \in, \Pi_X(A_1), ..., \Pi_X(A_l)).$$

$\Pi_X(\vec{a}) = \vec{a}$, $\Pi_X \restriction \kappa = id$.
If $y \in X \cap \mathcal{P}(\kappa)$, then $\Pi_X(y) = \{\Pi_X(t) : tEy\} = \{\Pi_X(t) : t \in y\} = y$. If $y \in N \cap \mathcal{P}(\kappa)$, there exists unique $x \in X$ such that $\Pi_X(x) = y$, and $x \in \mathcal{P}(\kappa)$, thus $\Pi_X(x) = x = y$. Therefore, $X \cap \mathcal{P}(\kappa) = N \cap \mathcal{P}(\kappa)$ and $\Pi_X(A_i) = A_i \cap N$.

$$(N, \in, \Pi_X(A_1), ..., \Pi_X(A_l)) \sqsubseteq (H_{\kappa^+}, \in, A_1, ..., A_l),$$

thus $(H_{\kappa^+}, \in, A_1, ..., A_l) \models \exists \vec{z}\phi(\vec{z}, \vec{a})$, i.e. $(H_{\kappa^+}, \in_{\Delta_1}^{H_{\kappa^+}}, A_1, ..., A_l) \models \exists \vec{x}\varphi(\vec{x}, \vec{a})$. □

Remark 3.3.1. *The standard Levy's Absoluteness Theorem is formulated in set theory as*

$$H_{\kappa^+} \preceq_1 V,$$

where κ is an infinite cardinal and $\preceq_1$ means that for any Σ_1 formula in set theory(Lévy Hierarchy) with parameters in H_{κ^+}, if it is true in V, then it is true in H_{κ^+}.
What we proved above is a slight more general case since A_i might not be in H_{κ^+}.
The reason we choose to prove Levy's Absoluteness Theorem in this form is that we need to add relation symbols to the language afterwards.

Corollary 3.3.1. *Let sets $B_i \subseteq \omega^\omega$ for each $i = 1, 2, ..., k$, then in language $\tau = \in_{\Delta_1} \cup \{B_1, ..., B_k\}$ we have*

$$(H_{\omega_1}, \in_{\Delta_1}^{H_{\omega_1}}, B_1, ..., B_k) \prec_1 (V, \in_{\Delta_1}^V, B_1, ..., B_k).$$

Proof. Since Δ_1 formulas in set theory are absolute for H_{ω_1}, $\omega^\omega \subseteq H_{\omega_1}$, we have

$$(H_{\omega_1}, \in_{\Delta_1}^{H_{\omega_1}}, B_1, ..., B_k) \sqsubseteq (V, \in_{\Delta_1}^V, B_1, ..., B_k).$$

For $f \in \omega^\omega$, let $C(f) = \{2^m \cdot 3^n : f(m) = n\}$. Then C is a 1-1 map from ω^ω to $\mathcal{P}(\omega)$.

For each B_i, define $A_i \subseteq \mathcal{P}(\omega)$ to be $A_i = \{C(f) : f \in B_i\}$.
let $\tau_0 = \in_{\Delta_1} \cup \{A_1, ..., A_k\}$, from Levy's Absoluteness Theorem,

$$(H_{\omega_1}, \in_{\Delta_1}^{H_{\omega_1}}, A_1, ..., A_k) \prec_1 (V, \in_{\Delta_1}^{V}, A_1, ..., A_k).$$

For any Σ_1 τ-formula $\phi(\vec{x})$, it is logically equivalent to the form $\exists\vec{y}\psi(\vec{x},\vec{y})$, where ψ is a quantifier-free τ-formula. Replace each $f \in B_i$ in ψ with $C(f) \in A_i$, we get a τ_0-formula χ. Since $Z = C(f)$ is Δ_1^0 in 2nd-order arithmetic, χ is $\Delta_1(ZFC^-_{\Delta_1})$ in language τ_0.
For any $\vec{a} \in (H_{\omega_1})^m$,

$$(H_{\omega_1}, \in_{\Delta_1}^{H_{\omega_1}}, A_1, ..., A_k) \models \exists\vec{y}\psi(\vec{a},\vec{y}) \leftrightarrow (H_{\omega_1}, \in_{\Delta_1}^{H_{\omega_1}}, B_1, ..., B_k) \models \exists\vec{y}\chi(\vec{a},\vec{y})$$

$$\leftrightarrow (V, \in_{\Delta_1}^{V}, B_1, ..., B_k) \models \exists\vec{y}\chi(\vec{a},\vec{y}) \leftrightarrow (V, \in_{\Delta_1}^{V}, A_1, ..., A_k) \models \exists\vec{y}\psi(\vec{a},\vec{y}).$$

$\square$

Remark 3.3.2. *Since a formula only involves finitely many A_i (or B_i), we can actually let language τ include arbitrary many relation symbols $A_i \subseteq \mathcal{P}(\kappa)$ (or $B_i \subseteq \omega^\omega$).*

We introduce another notation for substructures obtained by restricting the domain. After this section we will focus on models of set theory and apply Levy's Absoluteness several times, so it is necessary to introduce a simpler notation for this.

Definition 3.3.2. *For a τ-structure $\mathfrak{M}$ with domain M, and $N \subseteq M$, if $\mathfrak{M}$ restricted to N is a structure, than we say $N^{\mathfrak{M}}$ is*

$$N^{\mathfrak{M}} := (N; R_\lambda^{\mathfrak{M}} \cap N^{k_\lambda}, f_\beta^{\mathfrak{M}} \cap N^{k_\beta}, c_\gamma^{\mathfrak{M}}).$$

Remark 3.3.3. *Using this notation, if $\mathfrak{M} := (V, \in_{\Delta_1}^{V}, A_1, ..., A_l)$, here V is a set which we pretend to be our universe, then the conclusion of Levy's absoluteness theorem becomes*

$$H_{\kappa^+}^{\mathfrak{M}} \prec_1 \mathfrak{M},$$

where H_{κ^+} is the set $H_{\kappa^+}^{V}$.

We apply Shoenfield's Absoluteness Theorem and Levy's Absoluteness Theorem to get a nice property for forcing extension below. This proposition is formalized entirely in set theory.

Theorem 3.3.2. *If $V \models ZFC$, G is V-generic for some partial order $P \in V$, then*

$$V \equiv_1 V[G].$$

$\equiv_1$ here is for Lévy Hierarchy.

Proof. By Levy's Absoluteness Theorem (the version in set theory), we have

$$H_{\omega_1}^V \preceq_1 V,$$

$$H_{\omega_1}^{V[G]} \preceq_1 V[G].$$

From Theorem 3.1.2 and the remark below, we know for any Σ_1 sentence σ, there is a Σ_2^1 sentence σ_0 such that for any transitive $M \models ZFC$

$$H_{\omega_1}^M \models \sigma \Leftrightarrow M \models \sigma_0.$$

Since we know that any forcing extension doesn't add ordinals to the original one, so $\omega_1^{V[G]} \in V$.
From Shoenfield's Absoluteness Theorem, we have

$$V \models \sigma_0 \Leftrightarrow V[G] \models \sigma_0.$$

Then

$$H_{\omega_1}^V \models \sigma \Leftrightarrow V \models \sigma_0 \Leftrightarrow V[G] \models \sigma_0 \Leftrightarrow H_{\omega_1}^{V[G]} \models \sigma.$$

Thus, we checked that Σ_1 and Π_1 sentences are invariant through forcing extensions. $\square$

4 Model Companion of Set Theory

In this section, we want to explore the model companion of set theory.

However, we can not do it in the original language $\{\in\}$ since this language is too simple that the model companion of it(which only has the same universal part with set theory) would say much about set theory. We want the model companion to at least maintain the simple facts in set theory, e.g. the least countable ordinal, the axiom of extensionality and foundation... A natural candidate is the language $\in_{\Delta_0}$, which partially morleyize $\{\in\}$ to Δ_0 in set theory.

In any reasonable language τ expending $\in_{\Delta_0}$, the possible model companion of ZFC_τ(axioms of set theory in this language) is basically the theory of H_{ω_1}(or H_{κ^+} if we add a constant symbol κ and restrict it to be an infinite cardinal). Since model companion is model complete, every definable set in H_{ω_1} is Σ_1 in this language.

We will prove that in the language $\in_{\Delta_0}$, there is no model companion of ZFC_{Δ_0}, since in this language we can tell apart different complexity levels in H_{ω_1}. In order to solve this problem, we need to add more symbols to the language to increase its ability to express things so that definable set in H_{ω_1} will be Σ_1.

This section follows [Venturi and Viale, 2022a], [Venturi and Viale, 2022b] and [Viale, 2021].

4.1 In language $\in_{\Delta_0}$

We first prove that the model companion of ZFC_{Δ_0} satisfies $ZFC^-_{\Delta_0}$.

Lemma 4.1.1. *In a definable language $\tau \supseteq \in_{\Delta_0}$, if $\mathfrak{N} \models ZFC^-_\tau$ and $\mathfrak{M} \prec_1 \mathfrak{N}$, then*

$$\mathfrak{M} \models Z^-_{\Delta_0} + AC + \{Rep(\phi) : \phi \textit{ is a } \Sigma_1 \textit{ formula in } \tau\},$$

where $Rep(\phi)$ is Replacement Axiom for ϕ.

Proof. From Remark 2.4.4, $Z^-_{\Delta_0}$ is universal in $\in_{\Delta_0}$, thus $(ZFC^-_\tau)_{\Pi_2} \vdash Z^-_{\Delta_0}$. AC is equivalent to $\forall x \exists f \exists \alpha (f : x \to \alpha$ is bijection $\wedge\ \alpha \in Ord)$, which under Foundation is Π_2, thus $(ZFC^-_\tau)_{\Pi_2} \vdash AC$.
By Lemma 2.3.1, $\mathfrak{M} \models (ZFC^-_\tau)_{\Pi_2}$.
For Σ_1 formula ϕ in τ, let $\mathfrak{M} \models T_\forall$, there exists $\mathfrak{N}$ such that $\mathfrak{M} \sqsubseteq \mathfrak{N} \models T$. For any $a, \vec{b} \in |\mathfrak{M}|^{<\omega}$, if $\mathfrak{M} \models \forall x \in a(\exists! y \phi(x, y, \vec{b}))$, $\forall x \in a(\exists! y \phi(x, y, \vec{b}))$ is $\Sigma_1 \wedge \Pi_1$, $\mathfrak{M} \prec_1 \mathfrak{N}$, thus $\mathfrak{N} \models \forall x \in a(\exists! y \phi(x, y, \vec{b}))$.
Since $\mathfrak{N} \models Rep(\phi)$, $\mathfrak{N} \models \exists Y \forall x \in a \exists y \in Y \phi(x, y, \vec{b})$, thus $\mathfrak{M} \models \exists Y \forall x \in a \exists y \in Y \phi(x, y, \vec{b})$. □

Proposition 4.1.1. *In a definable language $\tau \supseteq \in_{\Delta_0}$, if $T \supseteq ZFC^-_\tau$ and T has a model companion S, then $S \supseteq ZFC^-_\tau$.*

This is Theorem 1.6(i) in [Viale, 2021].

Proof. For any $\mathfrak{M} \models S$, since S is model complete, $\mathfrak{M}$ is S-ec. Since $S_\forall = T_\forall$, there exists $\mathfrak{N} \models T$ such that $\mathfrak{M} \sqsubseteq \mathfrak{N}$. Since $\mathfrak{M}$ is S-ec, it is $S_\forall$-ec, and $\mathfrak{M} \prec_1 \mathfrak{N}$.
By Lemma 4.1.1, $\mathfrak{M} \models Z^-_{\Delta_0} + AC + \{Rep(\phi) : \phi$ is a Σ_1 formula in $\tau\}$, thus, $S \vdash Z^-_{\Delta_0} + AC + \{Rep(\phi) : \phi$ is a Σ_1 formula in $\tau\}$.
Since S is model complete, for any τ-formula ψ, there is a Σ_1-formula ϕ such that $S \vdash \psi \leftrightarrow \phi$, then $S \vdash Rep(\psi)$.
Thus, $S \supseteq ZFC^-_\tau$. □

We will show that in language $\in_{\Delta_0}$, any theory $T \supseteq ZFC_{\Delta_0}$ doesn't have a model companion.

Theorem 4.1.1. *In a definable language $\tau \supseteq \in_{\Delta_0}$, $T \supseteq ZFC_\tau$. If for any $(V, \in) \models ZFC$, $(H_{\omega_1}, \tau^{H_{\omega_1}}) \prec_1 (V, \tau^V)$, then*

$$\phi \equiv \forall x \exists f (f : \omega \to x \textit{ is surjective}) \in KH(T).$$

This is Theorem 1.6(ii) in [Viale, 2021].

Proof. Let ψ be a Π_2 formula in $\in_{\Delta_0}$ such that $Z^-_{\Delta_0} \vdash \phi \leftrightarrow \psi$.
For any $\mathfrak{M} \models T$, let $\mathfrak{M} = (V, \tau^V)$,

$$(H_{\omega_1}, \tau^{H_{\omega_1}}) \prec_1 (V, \tau^V).$$

Since $(H_{\omega_1}, \tau^{H_{\omega_1}}) \models \phi$ and $(H_{\omega_1}, \tau^{H_{\omega_1}}) \models Z^-_{\Delta_0}$, $Th(\mathfrak{M})_{\forall\vee\exists} + \psi$ is consistent.
By Lemma 2.3.7, $\psi \in KH(T)$. Since $Z^-_{\Delta_0} \subseteq KH(T)$, $\phi \in KH(T)$. □

Corollary 4.1.1. *In a definable language τ such that $\in_{\Delta_0} \subseteq \tau \subseteq \in_{\Delta_1}$,*

$$\forall x \exists f (f : \omega \to x \textit{ is surjective}) \in KH(ZFC_{\Delta_0}).$$

We can also use forcing to prove this corollary. The idea is that we can collapse any ordinal using forcing.

proof of Corollary 4.1.1 using forcing. Let $T = ZFC_{\Delta_0}$, for any T-ec $\mathfrak{M}$, we have $\mathfrak{N} = (V, \in^V_{\Delta_0})$ such that $\mathfrak{M} \prec_1 \mathfrak{N} \models T$.
Take $\lambda \in |\mathfrak{M}|$ such that $\mathfrak{M} \models \lambda$ is an ordinal, with Foundation, this statement is provably Δ_0 in language $\in_{\Delta_0}$, then

$$\mathfrak{N} \models \lambda \text{ is an ordinal.}$$

Since $\lambda \in V$, we take $\mathbb{P} = Coll(\lambda, \omega)$, where $Coll(\lambda, \omega)$ is the partial order on $\lambda^{<\omega}$ with $f \leq g$ iff $f \supseteq g$. See [Jech, 2006, Page 203, Example 14.3]. Then if $G \subset \mathbb{P}$ is generic over V, $(V[G], \in) \models \exists f (f : \omega \to \lambda \text{ is surjective})$,and

$$(V, \in^V_{\Delta_0}) \sqsubseteq (V[G], \in^{V[G]}_{\Delta_0}) \models ZFC_{\Delta_0}$$

since Δ_0 formulas in set theory are absolute for transitive models.
Since $\mathfrak{M}$ is T-ec, $\mathfrak{M} \prec_1 (V[G], \in^{V[G]}_{\Delta_0})$, then

$$\mathfrak{M} \models \exists f (f : \omega \to \lambda \text{ is surjective}).$$

Since $\mathfrak{M} \models AC$, $\mathfrak{M} \models \forall x \exists f (f : \omega \to x \text{ is surjective})$. $\square$

Now we know that in language $\in_{\Delta_0}$,

$$\forall x \exists f (f : \omega \to x \text{ is surjective}) \in KH(ZFC_{\Delta_0}).$$

Also we know that if a theory is model complete, it is axiomatized by its Π_2 part. A natural guess is that the Π_2 part of the theory of H_{ω_1} will be the model companion of ZFC_{Δ_0}. We'll prove this is wrong.

Theorem 4.1.2. *In language $\in_{\Delta_0}$, if theory $T \supseteq ZFC_{\Delta_0}$, then $S = \{\varphi : \varphi$ is a $\in_{\Delta_0}$-sentence and for any $\in_{\Delta_0}$-structure $\mathfrak{M} \models T$, $H^{\mathfrak{M}}_{\omega_1} \models \varphi\}$ is not model complete.*

This is Theorem 4.6 in [Venturi and Viale, 2022a].

Remark 4.1.1. *S is indeed a theory since it is closed under logical implication. In other words, $S = \{\varphi : T \models \varphi^{H_{\omega_1}}\}$.*

Proof. If S is model complete, any $\in_{\Delta_0}$-formula is S-equivalent to a Σ_1 formula, by Proposition 2.3.3. For a structure $\mathfrak{M} \models T$, $H^{\mathfrak{M}}_{\omega_1} \models S$, and every definable subset of $H^{\mathfrak{M}}_{\omega_1}$ with parameters is $\boldsymbol{\Sigma}_1$-definable.
We work inside $\mathfrak{M}$, and we know that $\mathfrak{M} \models ZFC_{\Delta_0}$. From Theorem 3.1.2 we know that every definable subset A of ω^ω is definable in $(H_{\omega_1}, \in)$, hence it is $\boldsymbol{\Sigma}_1$, i.e. it is $\boldsymbol{\Sigma}^1_2$.
However, from Corollary 2.1.1 we know that there is $B \in \boldsymbol{\Sigma}^1_2 \setminus \boldsymbol{\Pi}^1_2$, then $\omega^\omega \setminus B \in \boldsymbol{\Pi}^1_2 \setminus \boldsymbol{\Sigma}^1_2$, so that $\omega^\omega \setminus B$ is not $\boldsymbol{\Sigma}^1_2$, contradiction. $\square$

4.2 In a language extending $\in_{\Delta_0}$

In order to get a model companion of ZFC^-_τ, we want every definable set in H_{ω_1} to be $\mathbf{\Sigma}_1$. Since we can code every set in H_{ω_1} using a set in $\mathcal{P}(\omega^2)$(thus also coding using a set in $\mathcal{P}(\omega)$), we can add relation symbols for definable subsets of $\mathcal{P}(\omega)^k$ to the language. In order to get a model companion of ZFC^-_τ, we want every definable set in H_{ω_1} to be $\mathbf{\Sigma}_1$. Since we can code every set in H_{ω_1} using a set in $\mathcal{P}(\omega^2)$(thus also coding using a set in $\mathcal{P}(\omega)$), we can add relation symbols for definable subsets of $\mathcal{P}(\omega)^k$ to the language.

Definition 4.2.1. *We define formulas in 2nd-order arithmetic by recursion. We say a formulas is in 2nd-order arithmetic if:*
(1)it is logically equivalent to $\phi(\vec{x},\vec{Y}) \wedge (\bigwedge_i x_i \in \omega) \wedge (\bigwedge_i Y_i \subseteq \omega)$, where ϕ is a quantifier-free formula in set theory, or
(2)it is logically equivalent to $\exists z \in \omega\phi(z,\vec{x},\vec{Y})$, where ϕ is a formula in 2nd-order arithmetic, or
(3)it is logically equivalent to $\exists Z \subseteq \omega\phi(Z,\vec{x},\vec{Y})$, where ϕ is a formula in 2nd-order arithmetic, or
(4)it is logically equivalent to $\neg\phi$, where ϕ is a formula in 2md-order arithmetic, or
(5)it is logically equivalent to $\phi \wedge \psi$, where ϕ, ψ are formulas in 2md-order arithmetic.

Definition 4.2.2. *Given a formula $\phi(\vec{x},\vec{Y})$ in 2nd-order arithmetic, we say x_i is a 1st-order variable, and Y_j is a 2nd-order variable.*

Definition 4.2.3. *Define $\in_{arith}=\in_{\Delta_0} \cup\{R_\phi : \phi(\vec{X})$ is a formula in 2nd-order arithmetic and $\vec{X}$ are 2nd-order variables$\}$.*

Lemma 4.2.1. *In a definable language $\tau \supseteq\in_{\Delta_0}$, if theory S satisfies*
(1)$S \vdash ZFC^-_\tau$,
(2)$S \vdash \forall x\exists f(f : \omega \to x$ is surjective$)$,
(3)For each formula ϕ in set theory, there is a relation symbol R in language τ such that

$$S \vdash \forall\vec{x}(\phi^{H_{\omega_1}}(\vec{x}) \leftrightarrow \exists\vec{Y}[(\bigwedge_i(R_{Y_i} \in WF \wedge \Pi_{R_{Y_i}}(0) = x_i)) \wedge R(\vec{Y})]),$$

then S is model complete.

This is Theorem 3.4.8 in [Viale, 2021].

Proof. Since τ is a definable language, let A be the set of defining it. For a τ-formula $\varphi(\vec{x})$, replace any relation symbols and function symbols with the formula defining it, we get a formula $\phi_\varphi(\vec{x})$ in set theory and

$$P_A \vdash \varphi(\vec{x}) \leftrightarrow \phi_\varphi(\vec{x}).$$

Since $S \vdash ZFC^-_\tau + \forall x\exists f(f : \omega \to x \text{ is surjective})$, for any formula ϕ in set theory, we have

$$S \vdash \phi(\vec{x}) \leftrightarrow \phi^{H_{\omega_1}}(\vec{x}).$$

Combining with condition (3), there is a relation symbol R_φ in language τ such that
$$S \vdash \varphi(\vec{x}) \leftrightarrow \exists\vec{Y}[(\bigwedge_i \Pi_{R_{Y_i}}(0) = x_i) \wedge R_\varphi(\vec{x})].$$
Since $\Pi_{R_{Y_i}}(0) = x_i$ is $\Delta_1(ZFC^-)$ in set theory, $\exists\vec{Y}[(\bigwedge_i \Pi_{R_{Y_i}}(0) = x_i) \wedge R_\varphi(\vec{x})]$ is $\Sigma_1(S)$ in τ.
Thus, every τ-formula is T-equivalent to a Σ_1 τ-formula. $\square$

Theorem 4.2.1. *In language $\tau = \in_{arith}$, if theory $T \supseteq ZFC_\tau$, then $S = \{\varphi : \varphi$ is a τ-sentence and for any τ-structure $\mathfrak{M} \models T$, $H^{\mathfrak{M}}_{\omega_1} \models \varphi\}$ is the model companion of T.*

This is Theorem 5.2 in [Venturi and Viale, 2022a].

Remark 4.2.1. *It is easy to see that S is closed under logic implications, thus S is a theory. S is roughly the theory of H_{ω_1}.*

Proof. We first show that $T_{\forall\vee\exists} = S_{\forall\vee\exists}$. Roughly speaking, this is because H_{ω_1} and V have the same universal and existential parts.
For any τ-structure $\mathfrak{M} \models T$, from Proposition 3.3.1,
$$H^{\mathfrak{M}}_{\omega_1} \prec_1 \mathfrak{M}.$$
For any τ-sentence φ that is either Σ_1 or Π_1, if $\varphi \in T$, then for any $\mathfrak{M} \models T$, $H^{\mathfrak{M}}_{\omega_1} \models \varphi$, hence $\varphi \in S$. If $\varphi \in S$, for any $\mathfrak{M} \models T$, $H^{\mathfrak{M}}_{\omega_1} \models \varphi$, then $\mathfrak{M} \models \varphi$, hence $\varphi \in T$.
Thus, $T_{\forall\vee\exists} = S_{\forall\vee\exists}$. This implies $T_\forall = S_\forall$.
We need to check S satisfies the condition of Lemma 4.2.1.
From Proposition 3.3.1 and Proposition 3.3.1 we have for any $\mathfrak{M} \models T$, $H^{\mathfrak{M}}_{\omega_1} \models ZFC^-_{\Delta_0} + \forall x \exists f(f : \omega \to x$ is surjective).
Let $A = \{R_\phi : \phi(\vec{X})$ is a formula in 2nd-order arithmetic and $\vec{X}$ are 2nd-order variables$\}$, we need to show $H^{\mathfrak{M}}_{\omega_1} \models P_A$.
We can show by induction that for any formula ϕ in 2nd-order arithmetic,
$$ZFC \vdash \forall\vec{x}\forall\vec{Y}(\phi(\vec{x}, \vec{Y}) \leftrightarrow \phi^{H_{\omega_1}}(\vec{x}, \vec{Y})).$$
This is because $\omega = \omega^{H_{\omega_1}}$ and (f is a function)$\wedge$($f : \omega \to \omega$) are absolute for H_{ω_1}, V, and $\omega^\omega \subseteq H_{\omega_1}$.
Thus, $H^{\mathfrak{M}}_{\omega_1} \models P_A$.
We have checked condition (1)(2) in Lemma 4.2.1.
By Theorem 2.2.1 and Remark 2.2.5, we know that there is a relation symbol R in language τ such that
$$ZFC_\tau \vdash \forall\vec{x}(\phi^{H_{\omega_1}}(\vec{x}) \leftrightarrow \exists\vec{Y}[(\bigwedge_i (R_{Y_i} \in WF \wedge \Pi_{R_{Y_i}}(0) = x_i)) \wedge R(\vec{Y})]).$$
Since $R_{Y_i} \in WF$ and $\Pi_{R_{Y_i}}(0) = x_i$ are $\Delta_1(ZFC^-)$, there is a Σ_1 formula $\delta(\vec{x})$ in language τ such that
$$ZFC^-_\tau \vdash \delta(\vec{x}) \leftrightarrow \exists\vec{Y}[(\bigwedge_i (R_{Y_i} \in WF \wedge \Pi_{R_{Y_i}}(0) = x_i)) \wedge R(\vec{Y})].$$

For any $\mathfrak{M} \models T$, any $\vec{a} \in H_{\omega_1}^{\mathfrak{M}}$,

$$H_{\omega_1}^{\mathfrak{M}} \models \phi(\vec{a}) \Leftrightarrow \mathfrak{M} \models \delta(\vec{a}),$$

and from Levy's Absoluteness Theorem,

$$\mathfrak{M} \models \delta(\vec{a}) \Leftrightarrow H_{\omega_1}^{\mathfrak{M}} \models \delta(\vec{a}).$$

Thus, with $H_{\omega_1}^{\mathfrak{M}} \models ZFC_\tau^-$ and $H_{\omega_1}^{\mathfrak{M}} \models \forall \vec{x}[\phi(\vec{x}) \leftrightarrow \phi^{H_{\omega_1}}(\vec{x})]$,

$$H_{\omega_1}^{\mathfrak{M}} \models \forall \vec{x}(\phi^{H_{\omega_1}}(\vec{x}) \leftrightarrow \exists \vec{Y}[(\bigwedge_i (R_{Y_i} \in WF \wedge \Pi_{R_{Y_i}}(0) = x_i)) \wedge R(\vec{Y})]).$$

Thus, S satisfies the conditioin of Lemma 4.2.1, and S is model complete. $\square$

4.3 CH in model companion of set theory

In this section we want to investigate CH in the model companion of set theory in certain language. CH is the assertaion that $2^{\aleph_0} = \aleph_1$.

Note that without Power Set Axiom, there is no guarantee that ω_1 exists. If ω_1 does not exist, $\neg CH$ is vacuously true. For example, in language $\in_{arith}$, the model companion of set theory is the theory of H_{ω_1}, as we proved before. It turns out that H_{ω_1} asserts "there is no ω_1", as everything inside H_{ω_1} is countable and H_{ω_1} knows how to count them.

However, if we consider this in an extended language τ of $\in_{\Delta_0}$ and a theory $T \supseteq ZFC_\tau$, where we have a constant symbol ω_1 and a universal sentence "ω_1 is not countable", then it makes sense to consider the relation between CH and the model companion of T.

We need to mention here that our previous results on model companion of set theory can be easily extended to a language with an additional constant symbol κ and a theory extending set theory with an additional universal sentence "κ is an uncountable cardinal". Roughly speaking we can find an extended language with relation symbols for all definable subset of κ and in this language, the model companion of set theory exists, and is the theory of H_{κ^+}. For a proof, see [Viale, 2021].

Knowing that, the investigation over CH makes sense in this language with an additional constant symbol restricted to be an uncountable ordinal.

We first show that CH is provably Σ_2 under in any language extending $\in_{\Delta_0}$. This means $\neg CH$ is provably Π_2, which gives us a nice property since Π_2 sentence is preserved by T-ec structures.

Lemma 4.3.1. *CH is $\Sigma_2(Z_{\Delta_0}^- + AC)$ in language $\in_{\Delta_0}$.*

Proof. Under $Z^-_{\Delta_0}$, z is a cardinal $\equiv$

$$(z \text{ is an ordinal}) \wedge \forall \beta \in z \neg \exists g (g : \beta \to z \text{ bijection}),$$

which is $\Pi_1(Z^-_{\Delta_0} + AC)$ in $\in_{\Delta_0}$.
Under $Z^-_{\Delta_0} + AC$, $z = \omega_1 \equiv$

$$z \text{ is a cardinal} \wedge \omega \in z \wedge \exists F (F : \omega \times z \to z \wedge \forall \alpha \in z[\alpha \subseteq range(F \restriction \omega \times \{\alpha\})]),$$

which is $\Delta_2(Z^-_{\Delta_0} + AC)$ in $\in_{\Delta_0}$.
Under $Z^-_{\Delta_0}$, $CH \equiv$

$$\exists f[(f \text{ is a function}) \wedge (dom(f) = \omega_1) \wedge \forall p \subseteq \omega (p \in ran(f))],$$

where (f is a function), $(z = dom(f))$ are Δ_0, $\forall p \subseteq \omega (p \in ran(f))$ is Π_1.
Then CH is $\Sigma_2(Z^-_{\Delta_0} + AC)$ in $\in_{\Delta_0}$. □

Then we prove a theorem that investigates that relation between the Kaiser Hull of theories extending set theory and $\neg CH$.

Theorem 4.3.1. *In a definable language $\tau \supseteq \in_{\Delta_0}$, if $T \supseteq ZFC_\tau$, $T + \neg CH$ is consistent, and $KH(T)_{\forall\vee\exists} = T_{\forall\vee\exists}$, then $KH(T) + \neg CH$ is consistent.*

This is Proposition 3.5.1 in [Viale, 2021]. We modify the proof here since I believe there is an error in the original one.

Proof. Since $Z^-_{\Delta_0}$ is Π_2 axiomatizable, and $T_{\Pi_2} \subseteq KH(T)$, we have $Z^-_{\Delta_0} \subseteq KH(T)$.
From Lemma 4.3.1, we know that there is a Π_2 sentence σ such that

$$Z^-_{\Delta_0} + AC \vdash \neg CH \leftrightarrow \sigma.$$

Since $T + \neg CH$ is consistent, we have $T + \sigma$ is consistent, then $T_{\forall\vee\exists} + \sigma$ is consistent.
Since $KH(T)_{\forall\vee\exists} = T_{\forall\vee\exists}$, we have $KH(T)_{\forall\vee\exists} + \{\sigma\}_{\forall\vee\exists}$ is consistent.
We know that $KH(T)$ is axiomatize by Π_2 sentences, and σ is Π_2. Applying Proposition 2.3.4, $KH(T) + \sigma$ is consistent.
Since $Z^-_{\Delta_0} \subseteq KH(T)$, we have $KH(T) + \neg CH$ is consistent. □

Remark 4.3.1. *It is easy to see if we replace $\neg CH$ with any sentence which is $\Pi_2(Z^-_{\Delta_0} + AC)$, the theorem still holds.*

Remark 4.3.2. *The conditions of this theorem is natural, because we know that $ZFC + \neg CH$ is consistent(using forcing). $KH(T)_{\forall\vee\exists} = T_{\forall\vee\exists}$ is also a natural condition to pose since in a reasonable language τ with an additional constant symbol κ and theory $T \supseteq ZFC_\tau + \kappa$ is an uncountable cardinal, the model companion of T exists and is the theory of H_{κ^+}. From Levy's Absoluteness theorem, we have $H_{\kappa^+} \prec_1 V$, so H_{κ^+} and V agree on all universal and existential sentences.*

This theorem doesn't tell much about the relation between $\neg CH$ and $KH(T)$, since we know that $\neg CH$ is consistent with ZFC. The desirable result is to prove $\neg CH \in KH(T)$. To get this, we need to add another sentence asserting the existence of class many Woodin cardinals into theory T. For more details, see [Viale, 2021].

5 Conclusion

In conclusion, we've seen that the results in model theory can be applied to set theory, and we can take a model theoretic view point for results in set theory.

There are a bunch of things the thesis doesn't cover. In particular, in Corollary 4.1.1, the result is limited. We cannot apply it to $\in_{arith}$ that we later defined. And we didn't see how to use forcing to improve this result. If we consider definable language $\tau \supseteq \in_{\Delta_0}$ and $KH(T)$, since we need $V[G] \models T$, we need all defining formulas for symbols in τ to be absolute for $V, V[G]$. However, $(\omega^\omega)^{V[G]}$ might not be the same as $(\omega^\omega)^V$. We didn't solve this problem here.

And also, we didn't prove that in language $\in_{\Delta_0}$, if theory $T \supseteq ZFC_{\Delta_0}$, then it has no model companion. We proved a weaker result saying the theory of H_{ω_1} is not model complete, and we guess if the model companion exists, it is the theory of H_{ω_1}. We previously attempt to prove that the theory of H_{ω_1} is the Kaiser Hull of T. Then we encounter a technical problem asking us to prove Levy's Absoluteness Theorem inside ZFC^-. In Levy's Absoluteness Theorem, we need to use Löewenheim-Skolem theorem, which does not apply to proper classes. Thus, we only prove a restricted result saying the theory of H_{ω_1} is not model complete here.

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